Online Appendix for
"Realized Candlestick Wicks"

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This Online Appendix comprises three separate parts. Appendix A collects the proofs for all the theoretical results (Theorems and Corollaries) presented in the main text. Appendix B details the derivation of analytical moment results related to the normalized Brownian range and return. Appendix C contains additional results for both the Monte Carlo simulations (Section 4) and empirical applications (Section 5).

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Appendix A Proofs

In all the sequel, the positive constants $\{K, K', \dots\}$ may vary from line to line or even within a line, but never depend on n or on various indices. The symbol "$\asymp$" denotes the same order of magnitude, i.e., $f \asymp g \iff \exists K, K' > 0 : K|g| \leq |f| \leq K'|g|$.

A.1 Moments of Brownian Range and Return

For a standard Brownian motion starting at zero, i.e., $W = (W_t)_{t \geq 0}$, we denote the normalized high, low, and close as, respectively,

$$u = \sup_{0 \leq t \leq 1} W_t, \qquad d = \inf_{0 \leq t \leq 1} W_t, \qquad c = W_1. \tag{A.1}$$

The range of W over $[0, 1]$ is given by $\omega = u - d$. We further define the moments of $(\omega, |c|)$ as

$$\lambda_{p,r} = \mathbb{E}[\omega^p |c|^r]. \tag{A.2}$$

Table A.1 summarizes the analytical values of $\lambda_{p,r}$ for all combinations of $p, r \in \mathbb{N}$ such that $0 \leq p + r \leq 4$. Detailed derivations can be found in Appendix B.

Table A.1: Analytical values of $\lambda_{p,r}$

p \ r	0	1	2	3	4
0	1	$\sqrt{\frac{2}{\pi}}$	1	$2\sqrt{\frac{2}{\pi}}$	3
1	$2\sqrt{\frac{2}{\pi}}$	$\frac{3}{2}$	$\frac{8}{3}\sqrt{\frac{2}{\pi}}$	$\frac{15}{4}$	–
2	$4\ln 2$	$\frac{7}{9}\sqrt{\frac{\pi^3}{2}}$	$4\ln 2 + \frac{7}{4}\zeta(3)$	–	–
3	$\frac{2}{3}\sqrt{2\pi^3}$	$\frac{45}{8}\zeta(3)$	–	–	–
4	$9\zeta(3)$	–	–	–	–

Analytical values of joint moments of $(\omega, |c|)$, i.e., $\lambda_{p,r} = \mathbb{E}[\omega^p |c|^r]$ with $p, r \in \mathbb{N}$ and $0 \leq p + r \leq 4$, where ω (resp. c) is defined as the high-low range (resp. open-close return) of a standard Brownian motion within an unit interval.

A.2 Proof of Theorem 1

We prove Theorem 1 with the LLN for path-dependent functionals of continuous Itô semimartingales, as summarized in Duembgen and Podolskij (2015). Here we start with some notation. We denote by $C([0, 1])$ the space of continuous real-valued functions on the interval $[0, 1]$, and by $\|\cdot\|_\infty$ the supremum norm on $C([0, 1])$. A function $f : C([0, 1]) \to \mathbb{R}$ is said to have polynomial growth if $|f(x)| \leq C(1 + \|x\|_\infty^p)$ for some $C, p > 0$.

Definition A.1 (Local uniform continuity)**.** The function $f : C([0,1]) \to \mathbb{R}$ is locally uniformly continuous if for all $x \in C([0,1])$, there exists a closed ball of radius $K > 0$ centered at 0, i.e., $B_{\leq K}(0)$=$\{x \in C([0,1]); \|x\|_\infty \leq K\}$,[1] such that for every $\epsilon > 0$, there exists $\delta > 0$, for $x, y \in B_{\leq K}(0)$, $\|x-y\|_\infty \leq \delta$, we have $|f(x) - f(y)| \leq \epsilon$. This locally uniform continuity assumption is satisfied whenever $|f(x) - f(y)| \leq C\|x-y\|_\infty^p$ for all $x, y \in C([0,1])$ and some $C, p > 0$.

Lemma A.1 (Theorem 2.1, Duembgen and Podolskij, 2015)**.** Assume that the efficient price X follows a continuous Itô semimartingale in Eq. (5). Given a function $g : C([0,1]) \to \mathbb{R}$ and a vanishing sequence Δ_n, for the sequence of processes

$$\widehat{V}_{t,n}(g) = \Delta_n \sum_{i=1}^{n} g\left(\frac{d_i^n(X)}{\sqrt{\Delta_n}}\right), \tag{A.3}$$

where $d_i^n(X) = \{X_{(i-1+s)\Delta_n} - X_{(i-1)\Delta_n}; s \in [0,1]\}$, if g is locally uniformly continuous and has polynomial growth, it holds that

$$\widehat{V}_{t,n}(g) \xrightarrow{\text{u.c.p.}} V_t(g) = \int_0^t \rho_{\sigma_\tau}(g) d\tau, \tag{A.4}$$

as $n \to \infty$, where $\rho_z(g) = \mathbb{E}\left[g(\{zW_s; s \in [0,1]\})\right]$ whenever it is finite.

It is obvious that the WV estimator in Eq. (3) can be written in the form of Eq. (A.3) with a specific path-dependent function $g : C([0,1]) \to \mathbb{R}$ of the scaled incremental process:

$$\begin{aligned} g\left(\frac{d_i^n(X)}{\sqrt{\Delta_n}}\right) &= \frac{1}{4\ln 2 - 2}\left\{\sup_{0\leq s\leq 1} \frac{d_i^n(X)}{\sqrt{\Delta_n}} - \inf_{0\leq s\leq 1} \frac{d_i^n(X)}{\sqrt{\Delta_n}} - \frac{|X_{i\Delta_n} - X_{(i-1)\Delta_n}|}{\sqrt{\Delta_n}}\right\}^2 \\ &= \frac{1}{4\ln 2 - 2}\left\{f_1\left(\frac{d_i^n(X)}{\sqrt{\Delta_n}}\right) - f_2\left(\frac{d_i^n(X)}{\sqrt{\Delta_n}}\right)\right\}^2, \end{aligned} \tag{A.5}$$

where

$$f_1(x) = \sup_{0\leq s\leq 1} x(s) - \inf_{0\leq s\leq 1} x(s) \quad \text{and} \quad f_2(x) = |x(1) - x(0)|. \tag{A.6}$$

The function $g(x)$ is therefore a linear combination of polynomials of the range $f_1(x)$ and a finite power variation $f_2(x)$, as well as the cross term $f_1(x)f_2(x)$. This path-dependent function has polynomial growth, and is locally uniformly continuous. Then the LLN in Lemma A.1 readily

[1] The notion of locally uniform continuity is slightly different from the usual one that requires uniform continuity on neighborhoods or compact sets, see more details in Remark 2.1 in Duembgen and Podolskij (2015).

applies with

$$\begin{aligned}
\int_0^t \rho_{\sigma_\tau}(g) d\tau &= \int_0^t \mathbb{E}\left[g(\{\sigma_\tau W_s; s \in [0,1]\})\right] d\tau \\
&= \frac{1}{4\ln 2 - 2} \int_0^t \mathbb{E}\left[(f_1(\{\sigma_\tau W_s; s \in [0,1]\}) - f_2(\{\sigma_\tau W_s; s \in [0,1]\}))^2\right] d\tau \\
&= \frac{1}{4\ln 2 - 2} \int_0^t \mathbb{E}\left[\left(\sup_{0\le s\le 1} \sigma_\tau W_s - \inf_{0\le s\le 1} \sigma_\tau W_s - \sigma_\tau |W_1 - W_0|\right)^2\right] d\tau \\
&= \frac{1}{4\ln 2 - 2} \int_0^t \mathbb{E}\Bigg[\left(\sup_{0\le s\le 1} \sigma_\tau W_s - \inf_{0\le s\le 1} \sigma_\tau W_s\right)^2 + \sigma_\tau^2 W_1^2 \\
&\quad - 2\left(\sup_{0\le s\le 1} \sigma_\tau W_s - \inf_{0\le s\le 1} \sigma_\tau W_s\right)\sigma_\tau |W_1|\Bigg] d\tau \\
&= \frac{1}{4\ln 2 - 2} \int_0^t \sigma_\tau^2 \, \mathbb{E}\left[\omega^2 + c^2 - 2\omega|c|\right] d\tau \\
&= \frac{1}{4\ln 2 - 2} \int_0^t \sigma_\tau^2 \left(\lambda_{2,0} + \lambda_{0,2} - 2\lambda_{1,1}\right) d\tau \\
&= \frac{1}{4\ln 2 - 2} \int_0^t \sigma_\tau^2 \left(4\ln 2 + 1 - 2 \times \frac{3}{2}\right) d\tau \\
&= \int_0^t \sigma_\tau^2 d\tau,
\end{aligned} \tag{A.7}$$

where ω, c, and $\lambda_{p,r} = \mathbb{E}[\omega^p |c|^r]$ are defined in Appendix A.1. This completes the proof.

A.3 Proof of Theorem 2

We denote by $f'_y(x)$ the Gâteaux derivative of f at point x in the direction of y, i.e.,

$$f'_y(x) = \lim_{h\to 0} \frac{f(x+hy) - f(x)}{h}. \tag{A.8}$$

Lemma A.2 (Theorem 2.2, Duembgen and Podolskij, 2015)**.** Assume that the conditions of Lemma A.1 hold and Assumption 1 is satisfied. If $g'_y(x)$ for some $\|y\|_\infty \le 1$ is (i) locally uniformly continuous, and (ii) has polynomial growth, it follows that as $n \to \infty$,

$$\frac{1}{\sqrt{\Delta_n}} \left(\widehat{V}_{t,n}(g) - V_t(g)\right) \xrightarrow{\mathcal{L}-\mathrm{s}} U_t(g), \tag{A.9}$$

where $U_t(g) = \int_0^t u_\tau^{(1)} d\tau + \int_0^t u_\tau^{(2)} dW_\tau + \int_0^t u_\tau^{(3)} dW'_\tau$ with

$$\begin{aligned}
u_\tau^{(1)} &= \mu_\tau \rho_{\sigma_\tau}^{(2)}(g') + \frac{1}{2}\tilde{\sigma}_\tau \rho_{\sigma_\tau}^{(3)}(g') - \frac{1}{2}\tilde{\sigma}_\tau \rho_{\sigma_\tau}^{(2)}(g'), \\
u_\tau^{(2)} &= \rho_{\sigma_\tau}^{(1)}(g), \\
u_\tau^{(3)} &= \sqrt{\rho_{\sigma_\tau}(g^2) - \rho_{\sigma_\tau}^2(g) - (\rho_{\sigma_\tau}^{(1)}(g))^2},
\end{aligned} \tag{A.10}$$

and, for $z \in \mathbb{R}$ and $G(x, y) = g'_y(x)$,

$$\begin{aligned} \rho_z^{(1)}(g) &= \mathbb{E}[g(\{zW_s; s \in [0,1]\})W_1], \\ \rho_z^{(2)}(g') &= \mathbb{E}[G(\{zW_s; s \in [0,1]\}, \{s; s \in [0,1]\})], \\ \rho_z^{(3)}(g') &= \mathbb{E}[G(\{zW_s; s \in [0,1]\}, \{W_s^2; s \in [0,1]\})]. \end{aligned} \tag{A.11}$$

The process $W' = (W'_t)_{t \geq 0}$ is a Brownian motion defined on an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which is independent of $\mathcal{F}$. This is especially the case when g is an even function, i.e., $g(x) = g(-x)$ for all $x \in C([0,1])$, where it holds that

$$\rho_z^{(1)}(g) = \rho_z^{(2)}(g') = \rho_z^{(3)}(g') = 0, \tag{A.12}$$

for all $z \in \mathbb{R}$, since $W \stackrel{\mathcal{L}}{=} -W$ and expectations of odd functions of W are 0, and hence we have

$$U_t(g) = \int_0^t \sqrt{\rho_{\sigma_\tau}(g^2) - \rho_{\sigma_\tau}^2(g)}\, dW'_\tau. \tag{A.13}$$

which is an $\mathcal{F}$-conditional Gaussian martingale.

As mentioned in Appendix A.2, the path-dependent function $g : C([0,1]) \to \mathbb{R}$ in Eq. (A.5) is a linear combination of f_1^2, f_2^2, and $f_1 f_2$. Even though the stable CLT for $f_2^2(d_i^n(X)/\sqrt{\Delta_n})$ is easily deduced from Lemma A.2 (cf. Example 1 in Section 3, Duembgen and Podolskij, 2015), the result of g cannot be obtained straightforwardly because the range is not Gâteaux differentiable in general.

However, we may replace the Gâteaux derivative by an alternative form for functions which are not Gâteaux differentiable. We consider a range-based functional $\xi(x) = f(f_1(x))$ as an example, where $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, such that both f and f' have polynomial growth (cf. Example 3 in Section 3, Duembgen and Podolskij, 2015), by applying the following lemma from Christensen and Podolskij (2007):

Lemma A.3. Given two continuous functions $x, y \in C([0,1])$, assume t^* is the only point in $[0,1]$ where the maximum of x is achieved, i.e., $t^* = \operatorname{argmax}_{0\leq s\leq 1} x(s)$. Then it holds that

$$\lim_{h\to 0} \frac{\sup_{0\leq s\leq 1}(x(s) + hy(s)) - \sup_{0\leq s\leq 1} x(s)}{h} = y(t^*). \tag{A.14}$$

In the proofs $x(t)$ plays the role of the Brownian motion, which attains its maximum (resp. minimum) at a unique point almost surely. Let $\bar{t} = \operatorname{argmax}_{0\leq s\leq 1} W_s$ and $\underline{t} = \operatorname{argmin}_{0\leq s\leq 1} W_s$. Then Lemma A.2

remains valid when σ is everywhere invertible (Christensen and Podolskij, 2012) with

$$
\begin{aligned}
\rho_z^{(1)}(\xi) &= \mathbb{E}\left[f\left(z \left(\sup_{0\le s\le 1} W_s - \inf_{0\le s\le 1} W_s \right) \right) W_1 \right], \\
\rho_z^{(2)}(\xi') &= \mathbb{E}\left[f'\left(z \left(\sup_{0\le s\le 1} W_s - \inf_{0\le s\le 1} W_s \right) \right) \left(\bar{t} - \underline{t}\right) \right], \\
\rho_z^{(3)}(\xi') &= \mathbb{E}\left[f'\left(z \left(\sup_{0\le s\le 1} W_s - \inf_{0\le s\le 1} W_s \right) \right) \left(W_{\bar{t}}^2 - W_{\underline{t}}^2\right) \right],
\end{aligned}
\tag{A.15}
$$

which extends the asymptotic theory in Lemma A.2 to general functions of range.

Moreover, the derivative of the cross term $f_1 f_2$ is a linear combination of two separate components that include f_1' and f_2', respectively. It means that for the path-dependent function $g : C([0,1]) \to \mathbb{R}$ in Eq. (A.5) we can obtain the closed-form $\rho_z^{(1)}(g)$, $\rho_z^{(2)}(g')$, and $\rho_z^{(3)}(g')$ for all $z \in \mathbb{R}$, and Eq. (A.12) holds when g is an even function. Therefore, the stable CLT in Lemma A.2 holds with the limiting process $U_t(g)$ in Eq. (A.13), where the squared integrand is given by

$$
\begin{aligned}
\rho_{\sigma_\tau}(g^2) - \rho_{\sigma_\tau}^2(g) &= \mathbb{E}\left[g^2(\{\sigma_\tau W_s; s \in [0,1]\})\right] - \left(\mathbb{E}\left[g(\{\sigma_\tau W_s; s \in [0,1]\})\right]\right)^2 \\
&= \frac{1}{(4\ln 2 - 2)^2} \Bigg\{ \mathbb{E}\left[\left(f_1(\{\sigma_\tau W_s; s \in [0,1]\}) - f_2(\{\sigma_\tau W_s; s \in [0,1]\})\right)^4 \right] \\
&- \left(\mathbb{E}\left[\left(f_1(\{\sigma_\tau W_s; s \in [0,1]\}) - f_2(\{\sigma_\tau W_s; s \in [0,1]\})\right)^2 \right] \right)^2 \Bigg\} \\
&= \frac{1}{(4\ln 2 - 2)^2} \Bigg\{ \mathbb{E}\left[\left(\sup_{0\le s\le 1} \sigma_\tau W_s - \inf_{0\le s\le 1} \sigma_\tau W_s - \sigma_\tau |W_1| \right)^4 \right] \\
&- \left(\mathbb{E}\left[\left(\sup_{0\le s\le 1} \sigma_\tau W_s - \inf_{0\le s\le 1} \sigma_\tau W_s - \sigma_\tau |W_1| \right)^2 \right] \right)^2 \Bigg\} \\
&= \frac{\sigma_\tau^4}{(4\ln 2 - 2)^2} \left\{ \mathbb{E}\left[(\omega - |c|)^4 \right] - \left(\mathbb{E}\left[(\omega - |c|)^2 \right] \right)^2 \right\} \\
&= \frac{\sigma_\tau^4}{(4\ln 2 - 2)^2} \left\{ \mathbb{E}\left[\omega^4 - 4\omega^3|c| + 2\omega^2 c^2 - 4\omega|c|^3 + c^4 \right] - \left(\mathbb{E}\left[\omega^2 + c^2 - 2\omega|c| \right] \right)^2 \right\} \\
&= \frac{\sigma_\tau^4}{(4\ln 2 - 2)^2} \left\{ \lambda_{4,0} - 4\lambda_{3,1} + 6\lambda_{2,2} - 4\lambda_{1,3} + \lambda_{0,4} - (\lambda_{2,0} + \lambda_{0,2} - 2\lambda_{1,1})^2 \right\} \\
&= \frac{40\ln 2 - 16(\ln 2)^2 - 3\zeta(3) - 16}{(4\ln 2 - 2)^2} \sigma_\tau^4 = \Theta \sigma_\tau^4.
\end{aligned}
\tag{A.16}
$$

This completes the proof.

A.4 Proof of Corollary 1

The proof is analogous to that of Theorem 1. The WQ estimator in Eq. (9) can be written in the form of Eq. (A.3) with a locally uniformly continuous function $g^2 : C([0,1]) \to \mathbb{R}$ of the scaled incremental process:

$$g^2\left(\frac{d_i^n(X)}{\sqrt{\Delta_n}}\right) = \frac{1}{\Lambda_4}\left\{f_1\left(\frac{d_i^n(X)}{\sqrt{\Delta_n}}\right) - f_2\left(\frac{d_i^n(X)}{\sqrt{\Delta_n}}\right)\right\}^4. \tag{A.17}$$

Then the LLN in Lemma A.1 readily applies with

$$\begin{aligned}
\int_0^t \rho_{\sigma_\tau}(g^2)d\tau &= \int_0^t \mathbb{E}\left[g^2(\{\sigma_\tau W_s; s\in[0,1]\})\right]d\tau \\
&= \frac{1}{\Lambda_4}\int_0^t \mathbb{E}\left[(f_1(\{\sigma_\tau W_s; s\in[0,1]\}) - f_2(\{\sigma_\tau W_s; s\in[0,1]\}))^4\right]d\tau \\
&= \frac{1}{\Lambda_4}\int_0^t \mathbb{E}\left[\left(\sup_{0\le s\le 1}\sigma_\tau W_s - \inf_{0\le s\le 1}\sigma_\tau W_s - \sigma_\tau |W_1|\right)^4\right]d\tau \\
&= \frac{1}{\Lambda_4}\int_0^t \sigma_\tau^4\,\mathbb{E}\left[\omega^4 - 4\omega^3|c| + 2\omega^2c^2 - 4\omega|c|^3 + c^4\right]d\tau \\
&= \frac{1}{\Lambda_4}\int_0^t \sigma_\tau^4\left(\lambda_{4,0} - 4\lambda_{3,1} + 6\lambda_{2,2} - 4\lambda_{1,3} + \lambda_{0,4}\right)d\tau \\
&= \frac{1}{\Lambda_4}\int_0^t \sigma_\tau^4\left\{9\zeta(3) - 4\times\frac{45}{8}\zeta(3) + 6\left(4\ln 2 + \frac{7}{4}\zeta(3)\right) - 4\times\frac{15}{4} + 3\right\}d\tau \\
&= \frac{1}{\Lambda_4}\int_0^t \sigma_\tau^4\left(24\ln 2 - 12 - 3\zeta(3)\right)d\tau \\
&= \int_0^t \sigma_\tau^4 d\tau.
\end{aligned} \tag{A.18}$$

This completes the proof.

A.5 Proof of Theorem 3

Under Assumption 2, we consider the jump component of X in the following form, which is valid as the jumps are of finite variation:

$$J_t = \int_0^t \int_{\mathbb{R}} \delta(s,x)\underline{p}(ds,dx), \tag{A.19}$$

where $\delta(\omega,t,x)$ on $\Omega\times\mathbb{R}_+\times\mathbb{R}$ is predictable, $\underline{p}(dt,dx)$ is a Poisson random measure on $\mathbb{R}_+\times\mathbb{R}$ with a compensator $\underline{q}(dt,dx) = dt\otimes\lambda(dx)$, and λ is a σ-finite measure on $\mathbb{R}_+$. Moreover, we have

$$\lim_{u\to 0^+} u^r \int_{\{|f_m|\ge u\}} \lambda(dx) \le \int_{\{|f_m|\ge u\}} |f_m|^r \lambda(dx) < \infty, \tag{A.20}$$

which implies, as $u\to 0$,

$$\lambda(\{x : |f_m(x)|\ge u\}) \equiv \int_{\{|f_m|\ge u\}} \lambda(dx) = O(u^{-r}). \tag{A.21}$$

Note that the jumps in J are not necessarily of infinite activity, since Assumption 2 is trivially satisfied when there exist only finite-activity jumps.

We split the jumps into "big" and "small" ones by selecting a sequence (u_n) of positive real numbers satisfying:

$$\frac{u_n}{\sqrt{\Delta_n}} \to \infty \qquad \text{and} \qquad u_n \Delta_n^{\beta - 1/2} \to 0, \tag{A.22}$$

for any $0 < \beta \leq 1/2$. Then we rewrite the Itô semimartingale $X = X' + J$ in Eq. (12) as:

$$X_t = \underbrace{X_t'}_{\text{Eq. (5)}} + \underbrace{\int_0^t \int_{\{|\delta(s,x)| \geq u_n\}} \delta(s,x) \underline{p}(ds,dx)}_{\text{“Big” Jumps: } J_{1,t}^n} + \underbrace{\int_0^t \int_{\{|\delta(s,x)| < u_n\}} \delta(s,x) \underline{p}(ds,dx)}_{\text{“Small” Jumps: } J_{2,t}^n}, \tag{A.23}$$

where the component J is partitioned into two n-dependent processes J_1^n and J_2^n. We shall show that the presence of both jump processes does not affect the asymptotic distribution of the WV estimator.

We first demonstrate that the probability of having more than one "big" jump in any of the n intervals approaches zero. As discussed in Section 2.3 of Aït-Sahalia and Jacod (2009), log-price increments that exceed a cutoff level $\alpha \Delta_n^{\varpi}$, for some $\alpha > 0$ and $\varpi \in (0, 1/2)$, are mainly contributed by "big" jumps of size of order at least Δ_n^{ϖ}, and those increments mostly contain a single "big" jump under infill asymptotics. We validate the argument in Lemma A.4 below. Our selected cutoff level u_n satisfying Eq. (A.22) is a specific case with the parameter ϖ arbitrarily close to, but below 1/2. This "optimal" choice of ϖ separates all jumps that either prevail over or are diluted within Brownian increments, which is similar to some applications of jump truncation procedures in the literature, see, e.g., Liao and Todorov (2024).

Lemma A.4. For any $\varpi \in (0, 1/2)$, for the event $E_n = \{N_i \leq 1 \text{ for all } 1 \leq i \leq n\}$, it holds that

$$\lim_{n \to \infty} \mathbb{P}(E_n^{\complement}) = \lim_{n \to \infty} \mathbb{P}\left(\bigcup_{i=1}^n \{N_i > 1\} \right) = 0, \qquad \text{where } N_i = \sum_{s \in I_{n,i}} \mathbb{1}_{\{|\Delta X_s| \geq \alpha \Delta_n^{\varpi}\}}. \tag{A.24}$$

Proof. Similar to the Assumption (S-HON) of Jacod et al. (2019), we have Assumption 2 with $\tau_1 = \infty$ without loss of generality by a standard localization procedure, such that $|\delta(\omega, t, x)| \wedge 1 \leq f(x)$ holds uniformly on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$.

The random variable N_i counts the number of jumps in the i-th interval $I_{n,i}$ whose absolute sizes exceed the cutoff level. We consider the case with the threshold u_n in Eq. (A.22), as it captures the maximum possible number of jumps. This can be expressed in terms of the Poisson integral, see Eq. (2.1.11) of Jacod and Protter (2012):

$$N_i = \sum_{s \in I_{n,i}} \mathbb{1}_{\{|\Delta X_s| \geq u_n\}} = \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{\{|\delta(s,x)| \geq u_n\}} \underline{p}(ds,dx) \leq \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{\{|f(x)| \geq u_n\}} \underline{p}(ds,dx), \tag{A.25}$$

By the properties of the Poisson measure $\underline{p}$, the integral in the right side of Eq. (A.25) follows a

Poisson distribution with the intensity parameter

$$\lambda_i = \int_{(i-1)\Delta_n}^{i\Delta_n} ds \int_{\{|f(x)|\geq u_n\}} \lambda(dx) = \Delta_n \lambda(\{x : |f(x)| \geq u_n\}) \leq K\Delta_n u_n^{-r} = o(\sqrt{\Delta_n}), \tag{A.26}$$

which is implied by Eq. (A.21). From properties of the Poisson distribution we further deduce that

$$\mathbb{P}(N_i > 1) \leq \mathbb{P}\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \int_{\{|f(x)|\geq u_n\}} \underline{p}(ds, dx) > 1\right) = \frac{\lambda_i^2}{2} + o(\lambda_i^2) \leq K\Delta_n^2 u_n^{-2r}. \tag{A.27}$$

Therefore, with the union bound of probability, we have

$$\mathbb{P}\left(\bigcup_{i=1}^{n} \{N_i > 1\}\right) \leq \sum_{i=1}^{n} \mathbb{P}(N_i > 1) \leq K\Delta_n u_n^{-2r} = o(1). \tag{A.28}$$

This completes the proof. □

Next, we provide some estimates for the size of "small" jumps in each interval. Lemma A.5 below derives the required probabilistic bounds for the range of J_2^n over the i-th interval:

Lemma A.5. For the purely discontinuous process J_2^n defined in Eq. (A.23) under Assumption 2, with the sequence (u_n) of thresholds satisfying Eq. (A.22), it holds that

$$\begin{aligned} M_i^{(1)} &= \sup_{s\in I_{n,i}} \left|J_{2,s}^n - J_{2,(i-1)\Delta_n}^n\right| = O_p(\Delta_n u_n^{1-r}), \\ M_i^{(2)} &= \sup_{s\in I_{n,i}} \left|J_{2,s}^n - J_{2,(i-1)\Delta_n}^n\right|^2 = O_p(\Delta_n u_n^{2-r}). \end{aligned} \tag{A.29}$$

Proof. Similar to the proof of Lemma A.4, we follow Assumption (S-HON) of Jacod et al. (2019) by letting $\tau_1 = \infty$. We start with the notation for a local p-th order variation of "small" jumps, which resembles the first quantity in Eq. (2.1.35) of Jacod and Protter (2012): For some $p \geq 1$, we define

$$\widehat{\delta}_{p,i} = \frac{1}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} ds \int_{\{|\delta(s,x)|<u_n\}} |\delta(s,x)|^p \lambda(dx). \tag{A.30}$$

For all $1 \leq i \leq n$, it holds that

$$\begin{aligned} \mathbb{E}[\widehat{\delta}_{p,i}|\mathcal{F}_{(i-1)\Delta_n}] &\leq \frac{1}{\Delta_n}\mathbb{E}\left[\int_{(i-1)\Delta_n}^{i\Delta_n} ds \int_{\{|\delta(s,x)|<u_n\}} |\delta(s,x)|^p \lambda(dx) \middle| \mathcal{F}_{(i-1)\Delta_n}\right] \\ &\leq \frac{1}{\Delta_n}\mathbb{E}\left[\int_{(i-1)\Delta_n}^{i\Delta_n} ds \int_{\{|\delta(s,x)|<u_n\}} u_n^{p-r}|\delta(s,x)|^r \lambda(dx) \middle| \mathcal{F}_{(i-1)\Delta_n}\right] \\ &\leq u_n^{p-r} \int_{\mathbb{R}} |f(x)|^r \lambda(dx), \end{aligned} \tag{A.31}$$

since $\delta(\omega, t, x)$ is bounded by the deterministic function $f(x)$. Denote the integral as a constant

$C_r = \int_{\mathbb{R}} |f(x)|^r \lambda(dx)$, we have

$$\mathbb{E}[\widehat{\delta}_{p,i} | \mathcal{F}_{(i-1)\Delta_n}] \leq C_r u_n^{p-r}. \tag{A.32}$$

Similarly, for another conditional expectation $\mathbb{E}[\widehat{\delta}_{1,i}^p | \mathcal{F}_{(i-1)\Delta_n}]$, we have

$$\begin{aligned}
\mathbb{E}[\widehat{\delta}_{1,i}^p | \mathcal{F}_{(i-1)\Delta_n}] &= \frac{1}{\Delta_n^p} \mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} ds \int_{\{|\delta(s,x)|<u_n\}} |\delta(s,x)| \lambda(dx) \right)^p \middle| \mathcal{F}_{(i-1)\Delta_n} \right] \\
&\leq \frac{1}{\Delta_n^p} \mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} ds \int_{\{|\delta(s,x)|<u_n\}} u_n^{1-r} |\delta(s,x)|^r \lambda(dx) \right)^p \middle| \mathcal{F}_{(i-1)\Delta_n} \right] \\
&\leq u_n^{p(1-r)} \left(\int_{\mathbb{R}} |f(x)|^r \lambda(dx) \right)^p \\
&\leq C_r^p u_n^{p(1-r)}.
\end{aligned} \tag{A.33}$$

Then by Lemma 2.1.7 of Jacod and Protter (2012), with the bounds for both $\mathbb{E}[\widehat{\delta}_{p,i} | \mathcal{F}_{(i-1)\Delta_n}]$ and $\mathbb{E}[\widehat{\delta}_{1,i}^p | \mathcal{F}_{(i-1)\Delta_n}]$ in Eqs. (A.32) and (A.33), respectively, we have for all $p \geq 1$,

$$\begin{aligned}
\mathbb{E}[M_i^{(p)} | \mathcal{F}_{(i-1)\Delta_n}] &\leq K \left(\Delta_n \mathbb{E}[\widehat{\delta}_{p,i} | \mathcal{F}_{(i-1)\Delta_n}] + \Delta_n^p \mathbb{E}[\widehat{\delta}_{1,i}^p | \mathcal{F}_{(i-1)\Delta_n}] \right) \\
&\leq K \left(C_r \Delta_n u_n^{p-r} + C_r^p \Delta_n^p u_n^{p(1-r)} \right) \\
&\leq K' \Delta_n u_n^{p-r},
\end{aligned} \tag{A.34}$$

where the latter term $K C_r^p \Delta_n^p u_n^{p(1-r)}$ reduces to $K' \Delta_n u_n^{p-r}$ since $1 < 1 + \varpi(p-r) < p < p + p\varpi(1-r)$ for some ϖ slightly smaller than $1/2$. The desired results in Lemma A.5 follow from the law of iterated expectation and Markov's inequality.

□

We now examine the impact of both "big" and "small" jumps on WV. For simplicity, we start with some notation: For the sequence of semimartingales $X = X' + J_1^n + J_2^n$ in Eq. (A.23), we have

$$\begin{aligned}
w_i &= \sup_{t,s \in I_{n,i}} |X_t - X_s|, \qquad r_i = X_{i\Delta_n} - X_{(i-1)\Delta_n}, \\
w_i' &= \sup_{t,s \in I_{n,i}} |X_t' - X_s'|, \qquad r_i' = X'_{i\Delta_n} - X'_{(i-1)\Delta_n}, \\
\Delta_i^n J_1 &= J^n_{1,i\Delta_n} - J^n_{1,(i-1)\Delta_n}, \qquad \Delta_i^n J_2 = J^n_{2,i\Delta_n} - J^n_{2,(i-1)\Delta_n}.
\end{aligned} \tag{A.35}$$

Since the scaling factor Λ_2 is a constant which has no impact on the asymptotic order of the sum, it suffices to show that the absolute difference between (re-scaled) WVs from X and X' satisfies

$$\begin{aligned}\left|\sum_{i=1}^{n}(w_i-|r_i|)^2-\sum_{i=1}^{n}(w_i'-|r_i'|)^2\right| &\le \sum_{i=1}^{n}|(w_i-|r_i|)^2-(w_i'-|r_i'|)^2| \\ &= \sum_{i=1}^{n}\underbrace{|w_i-|r_i|+w_i'-|r_i'||}_{U_i^{(1)}}\cdot\underbrace{|w_i-|r_i|-w_i'+|r_i'||}_{U_i^{(2)}} \\ &= o_p(\sqrt{\Delta_n}).\end{aligned} \tag{A.36}$$

We define the set

$$\Gamma_n=\left\{1\le i\le n:\sup_{s\in I_{n,i}}|\Delta X_s|\ge u_n\right\},\quad \text{with } k_n=|\Gamma_n|, \tag{A.37}$$

where $|A|$ stands for the cardinality of set A. Conditional on the event E_n defined in Lemma A.4, which has probability approaching 1, each interval whose index belongs to Γ_n accommodates exactly one "big" jump. Under Assumption 2, the finite variation of $J_1^n+J_2^n$ over $[0,t]$ implies that

$$u_nk_n\le\sum_{0\le s\le t}|\Delta X_s|<\infty, \tag{A.38}$$

and thus $k_n\asymp u_n^{-1}=o(\sqrt{n})$. Moreover, $|\Gamma_n^{\complement}|=n-k_n\asymp n$ under infill asymptotics.

We decompose the sum in Eq. (A.36) into two complementary parts:

$$\sum_{i=1}^{n}U_i^{(1)}U_i^{(2)}=\sum_{i\in\Gamma_n}U_i^{(1)}U_i^{(2)}+\sum_{i\in\Gamma_n^{\complement}}U_i^{(1)}U_i^{(2)}. \tag{A.39}$$

For any $i\in\Gamma_n$, it follows from the triangle inequality that

$$\begin{aligned}w_i &= \sup_{t,s\in I_{n,i}}|X_t'+J_{1,t}^n+J_{2,t}^n-X_s'-J_{1,s}^n-J_{2,s}^n| \\ &\le w_i'+\sup_{t,s\in I_{n,i}}|J_{1,t}^n-J_{1,s}^n|+\sup_{t,s\in I_{n,i}}|J_{2,t}^n-J_{2,s}^n| \\ &\le w_i'+|\Delta_i^nJ_1|+2M_i^{(1)},\end{aligned} \tag{A.40}$$

where $\sup_{t,s\in I_{n,i}}|J_{1,t}^n-J_{1,s}^n|=|\Delta_i^nJ_1|$ since each infinitesimal interval mostly contains a single "big" jump, as shown in Lemma A.4, and also

$$|r_i|=|r_i'+\Delta_i^nJ_1+\Delta_i^nJ_2|\ge|\Delta_i^nJ_1|-|r_i'+\Delta_i^nJ_2|. \tag{A.41}$$

We have

$$w_i-|r_i|\le w_i'+2M_i^{(1)}+|r_i'+\Delta_i^nJ_2|\le w_i'+|r_i'|+3M_i^{(1)}, \tag{A.42}$$

where $|\Delta_i^n J_1|$ in the right side of both Eq. (A.40) and Eq. (A.41) are canceled, and thus

$$U_i^{(1)} \leq 2w_i' + 3M_i^{(1)} \qquad \text{and} \qquad U_i^{(2)} \leq 2|r_i'| + 3M_i^{(1)}, \tag{A.43}$$

with

$$\begin{aligned} &\mathbb{E}[(U_i^{(1)})^2|\mathcal{F}_{(i-1)\Delta_n}] \\ &= 4\mathbb{E}[(w_i')^2|\mathcal{F}_{(i-1)\Delta_n}] + 9\mathbb{E}[M_i^{(2)}|\mathcal{F}_{(i-1)\Delta_n}] + 12\mathbb{E}[w_i^{(c)}|\mathcal{F}_{(i-1)\Delta_n}]\mathbb{E}[M_i^{(1)}|\mathcal{F}_{(i-1)\Delta_n}] \\ &\leq K\Delta_n + K'\Delta_n u_n^{2-r} + K''\Delta_n^{3/2}u_n^{1-r} \leq K'''\Delta_n, \end{aligned} \tag{A.44}$$

and

$$\mathbb{E}[(U_i^{(2)})^2|\mathcal{F}_{(i-1)\Delta_n}] \leq \mathbb{E}[(U_i^{(1)})^2|\mathcal{F}_{(i-1)\Delta_n}] \leq K\Delta_n, \tag{A.45}$$

where w_i and $M_i^{(1)}$ are independent, and the Burkholder-Davis-Gundy inequality implies that

$$\begin{aligned} \mathbb{E}[(w_i')^p|\mathcal{F}_{(i-1)\Delta_n}] &\leq 2\mathbb{E}\left[\left(\sup_{s\in I_{n,i}} |X_s'| - X_{(i-1)\Delta_n}'\right)^p \middle| \mathcal{F}_{(i-1)\Delta_n}\right] \\ &\leq K_p\mathbb{E}\left[\left(\int_{(i-1)\Delta_n}^{i\Delta_n} \sigma_s^2 ds\right)^{\frac{p}{2}} \middle| \mathcal{F}_{(i-1)\Delta_n}\right] \leq K_p'\Delta_n^{\frac{p}{2}}, \end{aligned} \tag{A.46}$$

for all $1 \leq p < \infty$. By the Cauchy-Schwarz inequality,

$$\mathbb{E}[U_i^{(1)}U_i^{(2)}|\mathcal{F}_{(i-1)\Delta_n}] \leq \sqrt{\mathbb{E}[(U_i^{(1)})^2|\mathcal{F}_{(i-1)\Delta_n}]\mathbb{E}[(U_i^{(2)})^2|\mathcal{F}_{(i-1)\Delta_n}]} \leq K\Delta_n. \tag{A.47}$$

Therefore, we have

$$\sum_{i\in\Gamma_n} U_i^{(1)}U_i^{(2)} = O_p(k_n\Delta_n) = O_p(\Delta_n u_n^{-1}) = o_p(\sqrt{\Delta_n}). \tag{A.48}$$

Next, for all $i \in \Gamma_n^{\complement}$, by the triangle inequality,

$$w_i = \sup_{t,s\in I_{n,i}} |X_t' + J_{2,t}^n - X_s' - J_{2,s}^n| \leq w_i' + 2M_i^{(1)}, \tag{A.49}$$

$$|r_i| = |r_i' + \Delta_i^n J_2| \geq |r_i'| - |\Delta_i^n J_2| \geq |r_i'| - M_i^{(1)}. \tag{A.50}$$

We have $w_i - |r_i| \leq w_i' - |r_i'| + 3M_i^{(1)}$, and thus

$$U_i^{(1)} \leq 2w_i' + 3M_i^{(1)} \qquad \text{and} \qquad U_i^{(2)} \leq 3M_i^{(1)}, \tag{A.51}$$

$$U_i^{(1)}U_i^{(2)} = (2w_i' + 3M_i^{(1)})3M_i^{(1)} = 6w_i'M_i^{(1)} + 9M_i^{(2)}, \tag{A.52}$$

with

$$\begin{aligned}\mathbb{E}[U_i^{(1)}U_i^{(2)}|\mathcal{F}_{(i-1)\Delta_n}] &= 6\mathbb{E}[w_i'|\mathcal{F}_{(i-1)\Delta_n}]\mathbb{E}[M_i^{(1)}|\mathcal{F}_{(i-1)\Delta_n}] + 9\mathbb{E}[M_i^{(2)}|\mathcal{F}_{(i-1)\Delta_n}] \\ &\leq K\Delta_n^{3/2}u_n^{1-r} + K'\Delta_n u_n^{2-r} = O(\Delta_n u_n^{2-r}).\end{aligned} \tag{A.53}$$

Since it is obvious that $u_n \ll \Delta_n^{1/(4-2r)}$ by Eq. (A.22), we have

$$\sum_{i\in\Gamma_n^{\complement}} U_i^{(1)}U_i^{(2)} = O_p(u_n^{2-r}) = o_p(\sqrt{\Delta_n}). \tag{A.54}$$

The results in Eqs. (A.48) and (A.54) are sufficient for Eq. (A.36), which holds on E_n with $\mathbb{P}(E_n) \to 1$ by Lemma A.4. This completes the proof.

A.6 Proof of Theorem 4

(i) Gradual Jump. Under scenario (i), by the monotonicity of H_t, the location of τ plays no role in the following analysis. We assume $\tau = 0$ without loss of generality:

$$X_t = X_t' + H_t, \qquad \text{where } H_t = \int_0^t \frac{c_2}{s^\alpha}ds, \qquad \frac{1}{2} < \alpha < 1. \tag{A.55}$$

The increment of H_t over the i-th interval is given by

$$\Delta_i^n H = \int_{(i-1)\Delta_n}^{i\Delta_n} \frac{c_2}{s^\alpha}ds = \frac{c_2}{1-\alpha}\Delta_n^{1-\alpha}f_\alpha(i), \tag{A.56}$$

where $f_\theta(x) = x^{1-\theta} - (x-1)^{1-\theta}$ is a monotonically decreasing function over $[1,\infty)$ with $f_\theta(1) = 1$ and $\lim_{x\to\infty} f_\theta(x) = 0$ for all $0 < \theta < 1$. When Δ_n is sufficiently small, the equation

$$f_\alpha(x) = \frac{1-\alpha}{c_2}\zeta\Delta_n^{\alpha-1/2} \tag{A.57}$$

has a unique solution denoted by Π_n, for a constant $\zeta > 0$ that is large enough. The mean value theorem indicates $f_\alpha(\Pi_n) = (1-\alpha)(\Pi_n - \gamma)^{-\alpha}$ for some $0 < \gamma < 1$. We therefore have $\Pi_n \asymp \Delta_n^{(1/2\alpha)-1}$, and Π_n can be taken as integers without loss of generality.

The role of H is no smaller than the diffusion component over the first Π_n intervals, while it starts to be swamped by volatility from the next interval since its contribution vanishes in the limit, i.e., $\Delta_i^n H = o_p(\sqrt{\Delta_n})$ for any $\Pi_n + 1 \leq i \leq n$. With this result, we can make the following decomposition:

$$\left|\sum_{i=1}^{n}(w_i-|r_i|)^2-\sum_{i=1}^{n}(w_i'-|r_i'|)^2\right| \leq \underbrace{\left|\sum_{i=1}^{\Pi_n}(w_i-|r_i|)^2-\sum_{i=1}^{\Pi_n}(w_i'-|r_i'|)^2\right|}_{A_1} + \underbrace{\left|\sum_{i=\Pi_n+1}^{n}(w_i-|r_i|)^2-\sum_{i=\Pi_n+1}^{n}(w_i'-|r_i'|)^2\right|}_{A_2}, \tag{A.58}$$

where

$$A_1 \leq \sum_{i=1}^{\Pi_n} U_i^{(1)}U_i^{(2)} \qquad \text{and} \qquad A_2 \leq \sum_{i=\Pi_n+1}^{n} U_i^{(1)}U_i^{(2)}, \tag{A.59}$$

where we use the same notation $U_i^{(1)}$ and $U_i^{(2)}$ as defined in Eq. (A.36), with $X = X' + H$ in this section. Intuitively, A_1 corresponds to the bias induced by H within the "explosive zone", and A_2 corresponds to the bias stemming from the presence of H in the price increments not in the vicinity of τ. Therefore, it suffices to show that A_1 and A_2 are both $O_p(\Delta_n^{1/2\alpha})$.

For all $1 \leq i \leq \Pi_n$, since the monotonicity of H_t implies $\sup_{t,s\in I_{n,i}}|H_t - H_s| = |\Delta_i^n H|$, we have

$$w_i = \sup_{t,s\in I_{n,i}} |X_t' + H_t - X_s' - H_s| \leq w_i' + |\Delta_i^n H|, \tag{A.60}$$

$$|r_i| = |r_i' + \Delta_i^n H| \geq |\Delta_i^n H| - |r_i'|, \tag{A.61}$$

such that $w_i - |r_i| \leq w_i' + |r_i'|$, and thus

$$U_i^{(1)} \leq 2w_i' \qquad \text{and} \qquad U_i^{(2)} \leq 2|r_i'|, \tag{A.62}$$

Therefore, it holds that

$$A_1 \leq \sum_{i=1}^{\Pi_n} U_i^{(1)}U_i^{(2)} = O_p(\Pi_n\Delta_n) = O_p(\Delta_n^{1/2\alpha}). \tag{A.63}$$

Next, for all $\Pi_n + 1 \leq i \leq n$, we have both Eq. (A.60) and

$$|r_i| = |r_i' + \Delta_i^n H| \geq |r_i'| - |\Delta_i^n H|, \tag{A.64}$$

such that $w_i - |r_i| \leq w_i' - |r_i'| + 2|\Delta_i^n H|$, and thus

$$U_i^{(1)} \leq 2w_i' - 2|r_i'| + 2|\Delta_i^n H| \qquad \text{and} \qquad U_i^{(2)} \leq 2|\Delta_i^n H|. \tag{A.65}$$

We follow the same steps as Eqs. (46) and (47) in Andersen et al. (2023): The Taylor expansion of

$x^{1-\alpha}$ around $x = i$ provides the following approximation for large i:

$$(i-1)^{1-\alpha} = i^{1-\alpha} - (1-\alpha)i^{-\alpha} + o(i^{-\alpha}), \tag{A.66}$$

such that $f_\alpha(i) = (1-\alpha)i^{-\alpha} + o(i^{-\alpha})$. Therefore, we have

$$\sum_{i=\Pi_n+1}^{n} (\Delta_i^n H)^2 \leq K\Delta_n^{2(1-\alpha)} \sum_{i=\Pi_n+1}^{n} i^{-2\alpha} \asymp \Delta_n^{2(1-\alpha)} \Pi_n^{1-2\alpha} \int_1^{n\Pi_n^{-1}} s^{-2\alpha} ds = O_p(\Delta_n^{1/2\alpha}). \tag{A.67}$$

Moreover, by the Cauchy-Schwarz inequality,

$$\begin{aligned} \sum_{i=\Pi_n+1}^{n} (w_i' - |r_i'|)|\Delta_i^n H| &\leq \sum_{i=\Pi_n+1}^{n} w_i'|\Delta_i^n H| \\ &\leq \sqrt{\sum_{i=\Pi_n+1}^{n} (w_i')^2 \sum_{i=\Pi_n+1}^{n} (\Delta_i^n H)^2} = O_p(\sqrt{\Delta_n}\Delta_n^{1/4\alpha}), \end{aligned} \tag{A.68}$$

with the results in Eq. (A.67) and in Eq. (A.46) by the Burkholder-Davis-Gundy inequality. Therefore, we have

$$A_2 \leq \sum_{i=\Pi_n+1}^{n} U_i^{(1)} U_i^{(2)} \leq 4 \sum_{i=\Pi_n+1}^{n} (w_i' - |r_i'|)|\Delta_i^n H| + 4 \sum_{i=\Pi_n+1}^{n} (\Delta_i^n H)^2 = O_p(\Delta_n^{1/2\alpha}). \tag{A.69}$$

(ii) Flash Crash. Under scenario (ii), the V-shaped flash crash is a combination of two inverted gradual jumps, such that the results in scenario (i) hold for all intervals that do not contain τ. Consequently, it suffices to demonstrate that the additional bias from the specific V-interval is of order $O_p(\Delta_n^{2-2\alpha})$.

Suppose τ lies within the k_n-th interval I_{n,k_n}. Let

$$h_n = \frac{\tau - (k_n - 1)\Delta_n}{\Delta_n} \tag{A.70}$$

represent the fraction of the V-interval between τ and the left endpoint of I_{n,k_n}. The value of h_n lies within $[0, 1]$ by construction, and does not converge to a specific value unless a specific subsequence of Δ_n is chosen.

The increment of H over the V-interval I_{n,k_n} is given by

$$r_{k_n}^H = \int_{(k_n-1)\Delta_n}^{\tau} \frac{c_1}{s^\alpha} ds + \int_{\tau}^{k_n\Delta_n} \frac{c_2}{s^\alpha} ds = \frac{\Delta_n^{1-\alpha}}{1-\alpha}(c_1 h_n^{1-\alpha} + c_2(1-h_n)^{1-\alpha}), \tag{A.71}$$

and the corresponding range of H is

$$w_{k_n}^H = \sup_{t,s \in I_{n,k_n}} |H_t - H_s| = \frac{\Delta_n^{1-\alpha}}{1-\alpha}(|c_1 h_n^{1-\alpha}| \vee |c_2(1-h_n)^{1-\alpha}|), \tag{A.72}$$

Moreover, we have

$$\begin{aligned} U_{k_n}^{(1)} &= \big|w_{k_n} - |r_{k_n}| + w'_{k_n} - |r'_{k_n}|\big| \le w_{k_n} + w'_{k_n} \le w_{k_n}^H + 2w'_{k_n} = O_p(\Delta_n^{1-\alpha}), \\ U_{k_n}^{(2)} &= \big|w_{k_n} - |r_{k_n}| - w'_{k_n} + |r'_{k_n}|\big| \le w_{k_n}^H + |r_{k_n}^H| \le 2w_{k_n}^H = O_p(\Delta_n^{1-\alpha}). \end{aligned} \tag{A.73}$$

Therefore, we have the additional bias from the V-interval

$$(w_{k_n} - |r_{k_n}|)^2 - (w'_{k_n} - |r'_{k_n}|)^2 \le U_{k_n}^{(1)} U_{k_n}^{(2)} = O_p(\Delta_n^{2-2\alpha}). \tag{A.74}$$

This completes the proof.

A.7 Proof of Theorem 5

We define the truncation threshold for wick lengths as $v_n = \zeta\Delta_n^{\varpi}$ for some $\zeta > 0$ and $1/4 < \varpi < 1/2$, and denote the indicator function by $\mathbb{I}_i = \mathbb{1}_{\{w_i - |r_i| \le v_n\}}$. For the V-interval with the wick length of order $O_p(\Delta_n^{2-2\alpha})$, the truncated wick length satisfies

$$(w_{k_n} - |r_{k_n}|)\mathbb{I}_{k_n} = O_p(\Delta_n^{\varpi}) = o_p(\Delta_n^{1/4}), \tag{A.75}$$

which implies that the additional bias from the V-interval into the truncated WV is of order $o_p(\sqrt{\Delta_n})$, and becomes negligible under the normalization by $\sqrt{\Delta_n}$.

We then demonstrate that the wick truncation has asymptotically no effect on any other intervals in the absence of V-shape, even in the presence of any monotone explosive trends. We consider the event $E_n = \{\mathbb{I}_i = 1, \text{ for all } 1 \le i \le n\}$, under which the wick-truncated WV is equivalent to the original WV. Therefore, it suffices to show that $\mathbb{P}(E_n) \to 1$ under scenario (i) in Theorem 4, which is equivalent to show

$$\mathbb{P}(E_n^{\complement}) = \mathbb{P}\left(\max_{1\le i\le n}\{w_i - |r_i|\} \ge v_n\right) \to 0. \tag{A.76}$$

From our previous results in Eqs. (A.60) and (A.61), we have

$$w_i - |r_i| \le w'_i + |r'_i| \le 2w'_i, \tag{A.77}$$

for all $i \in \{1, 2, \ldots, n\}$. Therefore, we have

$$\begin{aligned} \mathbb{P}\left(\max_{1\le i\le n}\{w_i - |r_i|\} \ge v_n\right) &\le \mathbb{P}\left(2\max_{1\le i\le n} w'_i \ge v_n\right) \le K\Delta_n^{-p\varpi}\mathbb{E}\left[\left(\max_{1\le i\le n} w'_i\right)^p\right] \\ &\le K\Delta_n^{-p\varpi}\mathbb{E}\left[\sum_{i=1}^n (w'_i)^p\right] \le K'\Delta_n^{p/2-p\varpi-1}, \end{aligned} \tag{A.78}$$

where the bounds follow from Markov's inequality, the maximal inequality, and Eq. (A.46) by the Burkholder-Davis-Gundy inequality. Since the above result holds with arbitrarily large $p > 0$, for any $1/4 < \varpi < 1/2$, we can always pick $p > 2/(1 - 2\varpi)$ to ensure that the bound $K'\Delta_n^{p/2-p\varpi-1}$ has a positive exponent. This implies Eq. (A.76) as desired.

Furthermore, we show that the wick truncation has no effect on intervals that contain any types of jumps in Eq. (A.23). For any interval that could contains both "big" and "small" jumps, i.e., $i \in \Gamma_n$ as defined in Eq. (A.37), or only "small" jumps, i.e., $i \in \Gamma_n^{\complement}$, we have from Eq. (A.42):

$$w_i - |r_i| \leq 2w_i' + 3M_i^{(1)}. \tag{A.79}$$

Following the same steps as in Eq. (A.78), we have

$$\begin{aligned}
\mathbb{P}\left(\max_{1\leq i\leq n}\{w_i - |r_i|\} \geq v_n\right) &\leq \mathbb{P}\left(\max_{1\leq i\leq n}\{2w_i' + 3M_i^{(1)}\} \geq v_n\right) \\
&\leq K\Delta_n^{-p\varpi}\left(\mathbb{E}\left[\left(\max_{1\leq i\leq n} w_i'\right)^p\right] + \mathbb{E}\left[\left(\max_{1\leq i\leq n} M_i^{(1)}\right)^p\right]\right) \\
&\leq K\Delta_n^{-p\varpi}\left(\mathbb{E}\left[\sum_{i=1}^n (w_i')^p\right] + \mathbb{E}\left[\sum_{i=1}^n M_i^{(p)}\right]\right) \\
&\leq K'\Delta_n^{-p\varpi}(\Delta_n^{p/2-1} + u_n^{p-r}),
\end{aligned} \tag{A.80}$$

where $r \in [0,1)$ under Assumption 2, u_n is defined in Eq. (A.22), and the result for $M_i^{(p)}$ is from Eq. (A.34). Suppose $u_n \asymp \Delta_n^{\gamma}$ with γ arbitrarily close to $1/2$, we may further pick some $1/4 < \varpi < \gamma$ such that the exponents in the two terms above are both positive whenever $p > \max\{2/(1-2\varpi), r\gamma/(\gamma-\varpi)\}$. This completes the proof.

A.8 Proof of Theorem 6

As an analogous result to Theorems 4 and 5, the bias of WV due to the component H can be proved following the same steps with similar simplifying assumptions: There exists one persistent noise episode $[0,1]$, which is triggered by some ambiguous information arriving at time 0, and the funtion $g^{(1)}$ takes the form $g_{gj}^{(1)}$ in Eq. (26). The process $\epsilon_t^{(1)}$ in H_t only introduces extra randomness to the duration of persistent noise episode, which shall be harmlessly ignored.

As shown in Eqs. (56) and (57) in Andersen et al. (2023), there exists an asymptotic correspondence between the two models of episodic extreme return persistence, and they are equivalent with identical asymptotic analyses if we let $\beta = 1-\alpha$. The increment of H_t on the i-th interval is

$$\Delta_i^n H = f^{(1)}(\Delta X_0, \eta)((i-1)^{\beta} - i^{\beta})\Delta_n^{\beta} = \eta\Delta X_0(i^{\beta} - (i-1)^{\beta})\Delta_n^{\beta}, \tag{A.81}$$

where $f^{(1)}(\Delta X_0, \eta) = -\eta\Delta X_0$ with $\eta \in (0,1]$. With $\beta = 1-\alpha \in (0,1/2)$, the above persistent noise increment is equivalent to the drift burst increment in Eq. (A.56). The "big" jump ΔX_τ has no impact on WV, see Appendix A.5. The proof from here can proceed following the same steps as the proof of Theorems 4 and 5. When the function $g^{(1)} = g_{fc}^{(1)}$, the asymptotic effect of H depends on the smaller of the two parameters β_1 and β_2.

A.9 Proof of Theorem 7

In this section, we establish the asymptotic distribution for a broader class of KV estimators that nests $\mathcal{V}_{\mathcal{W}}$, which subsumes the results for all estimators with $\boldsymbol{\omega} \in \mathcal{W}$: We define a 3-by-1 weight vector $\boldsymbol{\omega} \in \mathbb{R}^3$, and then the estimator is given by

$$\mathrm{KV}_{t,n}(\boldsymbol{\omega}) = \sum_{i=1}^{n} \boldsymbol{\omega}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\theta}_i, \tag{A.82}$$

with the following asymptotic distribution under infill asymptotics:

Theorem 7-$\boldsymbol{\omega}$. Assume that the efficient price X follows a continuous Itô semimartingale in Eq. (5) with Assumption 1 satisfied. For all $\boldsymbol{\omega} \in \mathbb{R}^3$, it holds that

$$\frac{1}{\sqrt{\Delta_n}} \left(\mathrm{KV}_{t,n}(\boldsymbol{\omega}) - \boldsymbol{\omega}' \boldsymbol{\iota} \int_0^t \sigma_s^2 ds \right) \xrightarrow{\mathcal{L}-\mathrm{s}} \mathcal{MN} \left(0, \Theta(\boldsymbol{\omega}) \int_0^t \sigma_s^4 ds \right), \tag{A.83}$$

where $\boldsymbol{\iota} = (1,1,1)'$,

$$\Theta(\boldsymbol{\omega}) = \boldsymbol{\omega}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\Omega} \boldsymbol{\Sigma}^{-1} \boldsymbol{\omega} - \boldsymbol{\omega}' \boldsymbol{\iota} \boldsymbol{\iota}' \boldsymbol{\omega}, \tag{A.84}$$

and the symmetric matrix

$$\boldsymbol{\Omega} = \begin{pmatrix} \lambda_{4,0} & \lambda_{3,1} & \lambda_{2,2} \\ \bullet & \lambda_{2,2} & \lambda_{1,3} \\ \bullet & \bullet & \lambda_{0,4} \end{pmatrix}, \tag{A.85}$$

with all $\lambda_{p,r}$ specified in Table A.1.

Proof. It follows the same steps as the proofs of Theorems 1 and 2, with a general form of the path-dependent function $g : C([0,1]) \to \mathbb{R}$ of the scaled incremental process:

$$\mathrm{KV}_{t,n}(\boldsymbol{\omega}) = \Delta_n \sum_{i=1}^{n} g\left(\frac{d_i^n(X)}{\sqrt{\Delta_n}}; \boldsymbol{\omega} \right), \qquad g(x;\boldsymbol{\omega}) = \boldsymbol{\omega}' \boldsymbol{\Sigma}^{-1} \boldsymbol{f}(x), \qquad \boldsymbol{f}(x) = \begin{pmatrix} f_1^2(x) \\ f_2^2(x) \\ f_1(x) f_2(x) \end{pmatrix}, \tag{A.86}$$

where both the functions f_1 and f_2 are defined in Eq. (A.6). As a linear combination of all locally uniformly continuous functions with polynomial growth in $\boldsymbol{f}(x)$, the function $g(x;\boldsymbol{\omega})$ satisfies the conditions in both Lemmas A.1 and A.3. By Lemma A.1, we have

$$\mathrm{KV}_{t,n}(\boldsymbol{\omega}) \xrightarrow{\text{u.c.p.}} \int_0^t \rho_{\sigma_\tau}(g) d\tau = \boldsymbol{\omega}' \boldsymbol{\Sigma}^{-1} \boldsymbol{\Sigma} \boldsymbol{\iota} \int_0^t \sigma_\tau^2 d\tau = \boldsymbol{\omega}' \boldsymbol{\iota} \int_0^t \rho_{\sigma_\tau}(g) d\tau. \tag{A.87}$$

Then, by Lemmas A.2 and A.3, it holds that

$$\frac{1}{\sqrt{\Delta_n}} \left(\mathrm{KV}_{t,n}(\boldsymbol{\omega}) - \boldsymbol{\omega}' \boldsymbol{\iota} \int_0^t \sigma_s^2 ds \right) \xrightarrow{\mathcal{L}-\mathrm{s}} \int_0^t \sqrt{\rho_{\sigma_\tau}(g^2) - \rho_{\sigma_\tau}^2(g)} dW_\tau', \tag{A.88}$$

where W' is a Brownian motion defined on an extension of $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, which is independent of $\mathcal{F}$, and

$$\rho_{\sigma_\tau}(g^2) - \rho_{\sigma_\tau}^2(g) = \sigma_\tau^4(\boldsymbol{\omega}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega}\boldsymbol{\Sigma}^{-1}\boldsymbol{\omega} - \boldsymbol{\omega}'\boldsymbol{\iota}\boldsymbol{\iota}'\boldsymbol{\omega}), \tag{A.89}$$

where $\boldsymbol{\Omega} = \mathbb{E}[\boldsymbol{f}(\{W_s; s \in [0,1]\})\boldsymbol{f}'(\{W_s; s \in [0,1]\})]$. This completes the proof of Theorem 7-$\boldsymbol{\omega}$.

$\square$

As the result in Theorem 7-$\boldsymbol{\omega}$ holds for all $\boldsymbol{\omega} \in \mathbb{R}^3$ and the KV estimator is linear in $\boldsymbol{\omega}$, by the Cramér-Wold device, we have for any $\boldsymbol{\omega}_i, \boldsymbol{\omega}_j \in \mathbb{R}^3$,

$$\frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} \mathrm{KV}_{t,n}(\boldsymbol{\omega}_i) - \boldsymbol{\omega}_i'\boldsymbol{\iota} \int_0^t \sigma_s^2 ds \\ \mathrm{KV}_{t,n}(\boldsymbol{\omega}_j) - \boldsymbol{\omega}_j'\boldsymbol{\iota} \int_0^t \sigma_s^2 ds \end{pmatrix} \xrightarrow{\mathcal{L}-\mathrm{s}} \mathcal{MN}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Theta(\boldsymbol{\omega}_i) & \Gamma(\boldsymbol{\omega}_i, \boldsymbol{\omega}_j) \\ \bullet & \Theta(\boldsymbol{\omega}_j) \end{pmatrix} \int_0^t \sigma_s^4 ds \right), \tag{A.90}$$

where

$$\Gamma(\boldsymbol{\omega}_i, \boldsymbol{\omega}_j) = \boldsymbol{\omega}_i'\boldsymbol{\Sigma}^{-1}\boldsymbol{\Omega}\boldsymbol{\Sigma}^{-1}\boldsymbol{\omega}_j - \boldsymbol{\omega}_i'\boldsymbol{\iota}\boldsymbol{\iota}'\boldsymbol{\omega}_j. \tag{A.91}$$

We recall the space $\mathcal{W} = \{\boldsymbol{\omega} \in \mathbb{R}^3 : \boldsymbol{\omega}'\boldsymbol{\iota} = 1\}$ for $\boldsymbol{\iota} = (1,1,1)'$. Indicated by Theorem 7-$\boldsymbol{\omega}$, the restriction $\boldsymbol{\omega}'\boldsymbol{\iota} = 1$ ensures that all KV estimators in $\mathcal{V}_{\mathcal{W}}$ are consistent when X follows Eq. (5). Moreover, $\mathcal{V}_{\mathcal{W}}$ is closed under affine combination, such that for any $\boldsymbol{\omega}_i, \boldsymbol{\omega}_j \in \mathcal{W}$, we have

$$\rho\, \mathrm{KV}_{t,n}(\boldsymbol{\omega}_i) + (1-\rho)\, \mathrm{KV}_{t,n}(\boldsymbol{\omega}_j) \in \mathcal{V}_{\mathcal{W}}. \tag{A.92}$$

The OKV estimator in $\mathcal{V}_{\mathcal{W}}$ is defined by Eq. (A.87) with

$$\boldsymbol{\omega}^* = \underset{\boldsymbol{\omega} \in \mathcal{W}}{\mathrm{argmin}}\, \Theta(\boldsymbol{\omega}), \tag{A.93}$$

which can be easily solved by the method of Lagrange multipliers:

$$\boldsymbol{\omega}^* = \frac{\boldsymbol{\Sigma}\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}\boldsymbol{\iota}}{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}\boldsymbol{\iota}} \approx (1.7103, -0.7647, 0.0544)' \quad \text{and} \quad \Theta^* = \frac{1}{\boldsymbol{\iota}'\boldsymbol{\Sigma}\boldsymbol{\Omega}^{-1}\boldsymbol{\Sigma}\boldsymbol{\iota}} - 1 \approx 0.2594. \tag{A.94}$$

This completes the proof.

A.10 Proof of Corollary 2

Under $\mathbb{H}_0$, we start from the joint convergence in Eq. (A.90) by taking the two estimators to be WV and OKV, respectively:

$$\frac{1}{\sqrt{\Delta_n}} \begin{pmatrix} \mathrm{WV}_{t,n} - \int_0^t \sigma_s^2 ds \\ \mathrm{OKV}_{t,n} - \int_0^t \sigma_s^2 ds \end{pmatrix} \xrightarrow{\mathcal{L}-\mathrm{s}} \mathcal{MN}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \Theta & \Theta^* \\ \bullet & \Theta^* \end{pmatrix} \int_0^t \sigma_s^4 ds \right), \tag{A.95}$$

where the form of the asymptotic covariance arises from the variance optimality of OKV. Moreover, since we have $\Delta_n^{-1/2}(\overline{\mathrm{WV}}_{t,n} - \mathrm{WV}_{t,n}) = o_p(1)$ as the wick truncation is asymptotically irrelevant under $\mathbb{H}_0$ following the proof of Theorem 5, $\mathrm{WV}_{t,n}$ can be replaced with $\overline{\mathrm{WV}}_{t,n}$ in the above joint CLT. By the continuous mapping theorem, the consistency of the wick-truncated WQ estimator, and the stability of the convergence, we obtain

$$\frac{\mathrm{OKV}_{t,n} - \overline{\mathrm{WV}}_{t,n}}{\sqrt{\Delta_n \Xi\, \overline{\mathrm{WQ}}_{t,n}}} \xrightarrow{\mathcal{L}-\mathrm{s}} \mathcal{N}(0,1), \tag{A.96}$$

where $\Xi = \Theta - \Theta^*$. The above result implies $T_{t,n} \xrightarrow{\mathcal{L}} \chi_1^2$ under $\mathbb{H}_0$ and demonstrates that the Hausman test is correctly sized under infill asymptotics.

To derive the consistency of the test statistic under $\mathbb{H}_1$, recall that OKV is a linear combination of WV, RRV, and RV, as shown in Eq. (34). Under the jump alternative, since it is well-known (see, e.g., Christensen and Podolskij, 2012) that both RV and RRV consistently estimate the quadratic variation of X, whereas the (wick-truncated) WV estimates the IV of X, this difference diverges in $T_{t,n}$ after squaring and multiplying by n, which leads to the desired consistency result.

Under the explosive trend alternative, we utilize the following decomposition of OKV, where the approximated numerical values of the positive constants (k_1, k_2, k_3) are given in Eq. (34):

$$\mathrm{OKV}_{t,n} = k_1 \mathrm{WV}_{t,n} + k_2(\mathrm{RRV}_{t,n} - \lambda_{2,0}^{-1}\mathrm{RV}_{t,n}) + (k_2\lambda_{2,0}^{-1} - k_3)\mathrm{RV}_{t,n}. \tag{A.97}$$

It is worth noting that $\tilde{k}_3 = k_2\lambda_{2,0}^{-1} - k_3 \approx 0.1615 > 0$. Furthermore, the RRV-RV difference above has the following explicit form:

$$\mathrm{RRV}_{t,n} - \lambda_{2,0}^{-1}\mathrm{RV}_{t,n} = \lambda_{2,0}^{-1}\sum_{i=1}^{n}(w_i + |r_i|)(w_i - |r_i|), \tag{A.98}$$

which is also a function of wick length. We shall show the consistency of our test under the drift burst model under Assumption 3, as the proof for the persistent noise case is analogous. Following the proof of Theorem 4, we denote by $\mathrm{RRV}'_{t,n}$ and $\mathrm{RV}'_{t,n}$ the RRV and RV constructed from the continuous part X' of X, respectively. It holds that

$$\mathrm{RRV}_{t,n} - \lambda_{2,0}^{-1}\mathrm{RV}_{t,n} = \mathrm{RRV}'_{t,n} - \lambda_{2,0}^{-1}\mathrm{RV}'_{t,n} + O_p(\Delta_n^{1-\alpha+1/4\alpha}), \tag{A.99}$$

and the last estimate is of order $o_p(\Delta_n^{2-2\alpha})$ for all $1/2 < \alpha < 1$, which originates from intervals in the vicinity of τ. Also, by Theorem 3.1 of Laurent et al. (2024), RV under this alternative has the following asymptotic representation:

$$\mathrm{RV}_{t,n} = \mathrm{RV}'_{t,n} + \Delta_n^{2-2\alpha}\zeta^{\mu^2}_{(\alpha,2,0)}, \tag{A.100}$$

where $\zeta^{\mu^2}_{(\alpha,2,0)}$ is a positive constant depending only on the rate of drift explosion α, with the general expression provided in Eq. (9) of Laurent et al. (2024). In the absence of V-shapes, Eqs. (A.99), (A.100) and Theorem 4 implies that

$$\mathrm{OKV}_{t,n} = \underbrace{k_1 \mathrm{WV}'_{t,n} + k_2 \mathrm{RRV}'_{t,n} + k_3 \mathrm{RV}'_{t,n}}_{\mathrm{OKV}'_{t,n}} + \tilde{k}_3 \Delta_n^{2-2\alpha} \zeta^{\mu^2}_{(\alpha,2,0)} + o_p(\Delta_n^{2-2\alpha}), \tag{A.101}$$

which further indicates that

$$\frac{\mathrm{OKV}_{t,n} - \overline{\mathrm{WV}}_{t,n}}{\sqrt{\Delta_n \Xi\, \overline{\mathrm{WQ}}_{t,n}}} \xrightarrow{\mathcal{L}-\mathrm{s}} Z + \Delta_n^{3/2-2\alpha} \left(\frac{\tilde{k}_3 \zeta^{\mu^2}_{(\alpha,2,0)}}{\Xi \int_0^t \sigma_s^4 ds} + o_p(1) \right), \tag{A.102}$$

where $Z \sim \mathcal{N}(0,1)$. As the second term on the RHS of Eq. (A.102) diverges positively for $3/4 < \alpha < 1$, the consistency of the Hausman test follows. Adding V-shapes to the alternative further shifts the mean of the normal distribution positively, and does not alter the consistency of the test. This completes the proof.

Appendix B Normalized High, Low and Close

In this section, we calculate all $\lambda_{p,r}$ summarized in Table A.1, with some of these analytical values utilized in Appendix A.

For the range of a standard Brownian motion, i.e., $\omega = u - d$, its probability distribution was firstly proposed by Feller (1951) and its moment generating function was then derived by Parkinson (1980), i.e., for the r-th moment:

$$\mathbb{E}[\omega^r] = \frac{4}{\sqrt{\pi}}\left(1 - \frac{4}{2^r}\right) 2^{\frac{r}{2}} \Gamma\left(\frac{r+1}{2}\right) \zeta(r-1), \tag{B.1}$$

where $\Gamma(x)$ and $\zeta(x)$ are the Gamma and Riemann's zeta functions, respectively. In particular, we have

$$\lambda_{1,0} = \mathbb{E}[\omega] = 2\sqrt{\frac{2}{\pi}} \approx 1.5958, \tag{B.2}$$

$$\lambda_{2,0} = \mathbb{E}[\omega^2] = 4 \ln 2 \approx 2.7726, \tag{B.3}$$

$$\lambda_{3,0} = \mathbb{E}[\omega^3] = \frac{2}{3}\sqrt{2\pi^3} \approx 5.2499, \tag{B.4}$$

$$\lambda_{4,0} = \mathbb{E}[\omega^4] = 9\zeta(3) \approx 10.8185. \tag{B.5}$$

Also, Garman and Klass (1980) reveals the following fourth moments of the normalized high, low, close via the generating functions $\mathbb{E}[u^p d^q c^r]$:[2]

$$\mathbb{E}[u^4] = \mathbb{E}[d^4] = \mathbb{E}[c^4] = 3, \qquad \mathbb{E}[u^2c^2] = \mathbb{E}[d^2c^2] = 2, \tag{B.6}$$

$$\mathbb{E}[u^3c] = \mathbb{E}[d^3c] = 2.25, \qquad \mathbb{E}[uc^3] = \mathbb{E}[dc^3] = 1.5, \tag{B.7}$$

$$\mathbb{E}[u^2dc] = \mathbb{E}[ud^2c] = \frac{9}{4} - 2\ln 2 - \frac{7}{8}\zeta(3) \approx -0.1881, \tag{B.8}$$

$$\mathbb{E}[u^2d^2] = 3 - 4\ln 2 \approx 0.2274, \tag{B.9}$$

$$\mathbb{E}[udc^2] = 2 - 2\ln 2 - \frac{7}{8}\zeta(3) \approx -0.4381, \tag{B.10}$$

$$\mathbb{E}[ud^3] = \mathbb{E}[u^3d] = 3 - 3\ln 2 - \frac{9}{8}\zeta(3) \approx -0.4318, \tag{B.11}$$

where $\zeta(3) = \sum_{n=1}^{\infty} n^{-3} \approx 1.2021$. It is straightforward that

$$\lambda_{2,2} = \mathbb{E}[\omega^2c^2] = \mathbb{E}[(u-d)^2c^2] = \mathbb{E}[u^2c^2] + \mathbb{E}[d^2c^2] - 2\mathbb{E}[udc^2] = 4\ln 2 + \frac{7}{4}\zeta(3) \approx 4.8762. \tag{B.12}$$

When we substitute the normalized close c in above moments with its absolute value $|c|$, it is obvious that the values in Eqs. (B.7) and (B.8) do not follow from Garman and Klass (1980). Different from the Garman-Klass triple (u, d, c), Meilijson (2011) considers $(\tilde{u}, \tilde{d}, |c|)$ where $(\tilde{u}, \tilde{d}) = (u, d)$ if

[2] See Appendix C in Garman and Klass (1980).

$c \geq 0$ while $(\tilde{u}, \tilde{d}) = -(d, u)$ if $c < 0$, and derives the second and fourth moments as follows:

$$\mathbb{E}[\tilde{u}^2] = \frac{7}{4}, \quad \mathbb{E}[\tilde{d}^2] = \frac{1}{4}, \quad \mathbb{E}[\tilde{u}|c|] = \frac{5}{4}, \quad \mathbb{E}[\tilde{d}|c|] = -\frac{1}{4}, \quad \mathbb{E}[\tilde{u}\tilde{d}] = 1 - 2\ln 2 \approx -0.3863, \tag{B.13}$$

$$\mathbb{E}[\tilde{u}^4] = \frac{93}{16}, \quad \mathbb{E}[\tilde{d}^4] = \frac{3}{16}, \quad \mathbb{E}[\tilde{u}^2|c|^2] = \frac{31}{8}, \quad \mathbb{E}[\tilde{d}^2|c|^2] = \frac{1}{8}, \tag{B.14}$$

$$\mathbb{E}[\tilde{u}^3|c|] = \frac{147}{32}, \quad \mathbb{E}[\tilde{d}^3|c|] = -\frac{3}{32}, \quad \mathbb{E}[\tilde{u}|c|^3] = \frac{27}{8}, \quad \mathbb{E}[\tilde{d}|c|^3] = -\frac{3}{8}, \tag{B.15}$$

$$\mathbb{E}[\tilde{u}^2\tilde{d}^2] = \mathbb{E}[u^2 d^2] = 3 - 4\ln 2 \approx 0.2274, \tag{B.16}$$

$$\mathbb{E}[\tilde{u}\tilde{d}|c|^2] = \mathbb{E}[udc^2] = 2 - 2\ln 2 - \frac{7}{8}\zeta(3) \approx -0.4381, \tag{B.17}$$

$$\mathbb{E}[\tilde{u}^3\tilde{d}] + \mathbb{E}[\tilde{u}\tilde{d}^3] = \mathbb{E}[ud(u^2 + d^2)] = 6 - 6\ln 2 - \frac{9}{4}\zeta(3) \approx -0.8635, \tag{B.18}$$

$$\mathbb{E}[\tilde{u}^2\tilde{d}|c|] + \mathbb{E}[\tilde{u}\tilde{d}^2|c|] = \mathbb{E}[udc(u+d)] = \frac{9}{2} - 4\ln 2 - \frac{7}{4}\zeta(3) \approx -0.3762, \tag{B.19}$$

$$\mathbb{E}[\tilde{u}\tilde{d}^2|c|] = \frac{1}{16}\zeta(3) - 2\ln 2 + \frac{47}{32} \approx 0.1576. \tag{B.20}$$

We can use the above results to obtain the following second and fourth moments of $(\omega, |c|)$:

$$\lambda_{1,1} = \mathbb{E}[\omega|c|] = \mathbb{E}[(\tilde{u} - \tilde{d})|c|] = \mathbb{E}[\tilde{u}|c|] - \mathbb{E}[\tilde{d}|c|] = \frac{3}{2}, \tag{B.21}$$

$$\lambda_{1,3} = \mathbb{E}[\omega|c|^3] = \mathbb{E}[(\tilde{u} - \tilde{d})|c|^3] = \mathbb{E}[\tilde{u}|c|^3] - \mathbb{E}[\tilde{d}|c|^3] = \frac{15}{4}, \tag{B.22}$$

$$\begin{aligned}
\lambda_{3,1} &= \mathbb{E}[\omega^3|c|] = \mathbb{E}[(\tilde{u} - \tilde{d})^3|c|] \\
&= \mathbb{E}[(\tilde{u}^3 - \tilde{d}^3 - 3\tilde{u}^2\tilde{d} + 3\tilde{u}\tilde{d}^2)|c|] \\
&= \mathbb{E}[\tilde{u}^3|c|] - \mathbb{E}[\tilde{d}^3|c|] - 3\mathbb{E}[\tilde{u}^2\tilde{d}|c|] + 3\mathbb{E}[\tilde{u}\tilde{d}^2|c|] \\
&= \frac{147}{32} + \frac{3}{32} - 3\left(\frac{9}{2} - 4\ln 2 - \frac{7}{4}\zeta(3)\right) + 6\left(\frac{1}{16}\zeta(3) - 2\ln 2 + \frac{47}{32}\right) \\
&= \frac{45}{8}\zeta(3) \approx 6.7616.
\end{aligned} \tag{B.23}$$

To calculate the third moments of $(\omega, |c|)$, we derive the analytical expressions for $\mathbb{E}[\tilde{u}^2|c|]$, $\mathbb{E}[\tilde{d}^2|c|]$, $\mathbb{E}[\tilde{u}|c|^2]$, $\mathbb{E}[\tilde{d}|c|^2]$, and $\mathbb{E}[\tilde{u}\tilde{d}|c|]$, which are not available in the literature. For the first four quantities, we obtain the results by integrating the joint densities in Meilijson (2011), i.e.,

$$f_{\tilde{u},|c|}(a, x) = 4(2a - x)\phi(2a - x), \quad 0 < x < a, \tag{B.24}$$

$$f_{\tilde{d},|c|}(b, x) = 4(x - 2b)\phi(x - 2b), \quad b < 0 < x, \tag{B.25}$$

where $\phi(z) = (2\pi)^{-1/2}e^{-z^2/2}$ is the probability density function (PDF) of $\mathcal{N}(0, 1)$:

$$\mathbb{E}[\tilde{u}^2|c|] = \frac{17}{3\sqrt{2\pi}} \approx 2.2607, \quad \mathbb{E}[\tilde{d}^2|c|] = \frac{1}{3\sqrt{2\pi}} \approx 0.1330, \tag{B.26}$$

$$\mathbb{E}[\tilde{u}|c|^2] = \frac{7}{3}\sqrt{\frac{2}{\pi}} \approx 1.8617, \quad \mathbb{E}[\tilde{d}|c|^2] = -\frac{1}{3}\sqrt{\frac{2}{\pi}} \approx -0.2660. \tag{B.27}$$

There is one more moment needed, i.e., $\mathbb{E}[\tilde{u}\tilde{d}|c|]$. We start with the infinitesimal event $A = \{W_1 \in (x, x+dx), W_t \in (b,a), \forall t \in [0,1]\}$, where $b < \min\{x,0\} \leq 0 \leq \max\{x,0\} < a$, and its probability $\mathbb{P}(A) = Q(a,b,x)dx$, where

$$Q(a,b,x) = \sum_{j=-\infty}^{\infty} \left\{\phi(x - 2j(a-b)) - \phi(x - 2b - 2j(a-b))\right\}. \tag{B.28}$$

The joint density of $(\tilde{u}, \tilde{d}, |c|)$ is then given by $f_{\tilde{u},\tilde{d},|c|}(a,b,x) = -2\partial^2 Q(a,b,x)/\partial a \partial b$, restricted to $b < 0 < x < a$, which is also an infinite series.[3] The summand with $j = 0$ takes value 0 because both two ϕ functions are independent of at least one of a and b, as similar to the second term in the summand with $j = 1$. The required moment can be obtained by solving the triple integral:

$$\begin{aligned}
\mathbb{E}[\tilde{u}\tilde{d}|c|] &= -2\int_0^\infty \int_0^a \int_{-\infty}^0 abx\frac{\partial^2 Q(a,b,x)}{\partial a \partial b} db dx da \\
&= -2\sum_{j \in \complement_{\mathbb{Z}}\{0\}} \int_0^\infty a da \int_0^a x dx \int_{-\infty}^0 \frac{\partial}{\partial a} b \left[\frac{\partial}{\partial b}\phi(x - 2j(a-b)) \right. \\
&\quad \left. - \frac{\partial}{\partial b}\phi(x - 2b - 2j(a-b))\mathbb{1}_{\{j \neq 1\}}\right] db
\end{aligned} \tag{B.29}$$

We integrate each summand in three univariate steps. The first step will integrate over $b \in (-\infty, 0)$ the product of b and mixed second derivative $\partial^2\phi(x + Ma + Kb)/\partial a \partial b$:

$$\begin{aligned}
\int_{-\infty}^0 \frac{\partial}{\partial a} b \frac{\partial}{\partial b}\phi(x + Ma + Kb) db &= \frac{\partial}{\partial a}\int_{-\infty}^0 b\frac{\partial}{\partial b}\phi(x + Ma + Kb) db \\
&= \frac{\partial}{\partial a}\int_{-\infty}^0 b d\phi(x + Ma + Kb) \\
&= \frac{\partial}{\partial a}\left[b\phi(x + Ma + Kb)\right]_{-\infty}^0 - \frac{\partial}{\partial a}\int_{-\infty}^0 \phi(x + Ma + Kb) db \\
&= -\int_{-\infty}^0 \frac{\partial}{\partial a}\phi(x + Ma + Kb) db \\
&= -M\int_{-\infty}^0 \phi'(x + Ma + Kb) db \\
&= -\frac{M}{K}\left[\phi(x + Ma + Kb)\right]_{-\infty}^0 \\
&= -\frac{M}{K}\phi(x + Ma).
\end{aligned} \tag{B.30}$$

[3] See more details in the Appendix of Meilijson (2011).

Then we multiply the above result by x and integrate it over $x \in (0, a)$:

$$
\begin{aligned}
&\int_0^a x dx \int_{-\infty}^0 \frac{\partial}{\partial a} b \frac{\partial}{\partial b} \phi(x + Ma + Kb) db \\
=& -\frac{M}{K} \int_0^a x\phi(x + Ma) dx \\
=& -\frac{M}{K} \int_{Ma}^{(M+1)a} y\phi(y) dy + \frac{M^2 a}{K} \int_{Ma}^{(M+1)a} \phi(y) dy \\
=& \frac{M}{K} \int_{Ma}^{(M+1)a} \phi'(y) dy + \frac{M^2 a}{K} \int_{Ma}^{(M+1)a} \phi(y) dy \qquad \text{because } \phi'(z) = -z\phi(z) \\
=& \frac{M}{K} \left[\phi((M+1)a) - \phi(Ma)\right] + \frac{M^2 a}{K} \left[\Phi((M+1)a) - \Phi(Ma)\right] \\
\text{or } =& \frac{M}{K} \left[\phi((M+1)a) - \phi(Ma)\right] - \frac{M^2 a}{K} \left[\Phi^*((M+1)a) - \Phi^*(Ma)\right],
\end{aligned}
\tag{B.31}
$$

where $\Phi(z) = \int_{-\infty}^z \phi(t)dt = 0.5(1 + \operatorname{erf} z/\sqrt{2})$ is the cumulative distribution function (CDF) of $\mathcal{N}(0, 1)$, and $\Phi^*(z) = 1 - \Phi(z) = 0.5(1 - \operatorname{erf} z/\sqrt{2})$ is the survival function. Finally, this expression is multiplied by a and integrated over $a \in (0, \infty)$. We use the results

$$
\int_0^\infty a\phi(aA) da = \int_0^\infty a\phi(-aA) da = \frac{1}{\sqrt{2\pi} A^2},
\tag{B.32}
$$

$$
\int_0^\infty a^2 \Phi(-aA) da = \int_0^\infty a^2 \Phi^*(aA) da = \frac{1}{3A^3}\sqrt{\frac{2}{\pi}}, \quad \text{with } A > 0,
\tag{B.33}
$$

to calculate the triple integral of $abx\partial^2\phi(x + Ma + Kb)/\partial a\partial b$. When $M \in \mathbb{Z}^{>0}$, we have

$$
\begin{aligned}
&\int_0^\infty a da \int_0^a x dx \int_{-\infty}^0 \frac{\partial}{\partial a} b \frac{\partial}{\partial b} \phi(x + Ma + Kb) db \\
=& \frac{M}{K} \int_0^\infty a\phi((M+1)a) da - \frac{M}{K} \int_0^\infty a\phi(Ma) da + \frac{M^2}{K} \int_0^\infty a^2 \Phi((M+1)a) da - \frac{M^2}{K} \int_0^\infty a^2 \Phi(Ma) da \\
=& \frac{1}{\sqrt{2\pi}} \frac{M}{K} \left[\frac{1}{(M+1)^2} - \frac{1}{M^2}\right] - \frac{1}{3}\sqrt{\frac{2}{\pi}} \frac{M^2}{K} \left[\frac{1}{(M+1)^3} - \frac{1}{M^3}\right] = \mathcal{G}(M, K).
\end{aligned}
\tag{B.34}
$$

When $M \in \mathbb{Z}^{<-1}$, we have

$$
\begin{aligned}
&\int_0^\infty a da \int_0^a x dx \int_{-\infty}^0 \frac{\partial}{\partial a} b \frac{\partial}{\partial b} \phi(x + Ma + Kb) db \\
=& \frac{M}{K} \int_0^\infty a\phi((M+1)a) da - \frac{M}{K} \int_0^\infty a\phi(Ma) da - \frac{M^2}{K} \int_0^\infty a^2 \Phi^*((M+1)a) da + \frac{M^2}{K} \int_0^\infty a^2 \Phi^*(Ma) da \\
=& \frac{1}{\sqrt{2\pi}} \frac{M}{K} \left[\frac{1}{(M+1)^2} - \frac{1}{M^2}\right] - \frac{1}{3}\sqrt{\frac{2}{\pi}} \frac{M^2}{K} \left[\frac{1}{(M+1)^3} - \frac{1}{M^3}\right] = \mathcal{G}(M, K).
\end{aligned}
\tag{B.35}
$$

We now transfer each summand into a rational function of j, by letting M take $-2j$, and K take $2j$ or $2(j-1)$. For summands with $j \in \complement_{\mathbb{Z}}\{0,1\}$, we have

$$
\begin{aligned}
&\int_0^\infty ada \int_0^a xdx \int_{-\infty}^0 \frac{\partial}{\partial a} b \left[\frac{\partial}{\partial b}\phi(x-2j(a-b)) - \frac{\partial}{\partial b}\phi(x-2b-2j(a-b))\right] db \\
&= \mathcal{G}(-2j,2j) - \mathcal{G}(-2j,2(j-1)) \\
&= -\frac{1}{\sqrt{2\pi}}\left(1-\frac{j}{j-1}\right)\left[\frac{1}{(1-2j)^2}-\frac{1}{(2j)^2}\right] - \frac{1}{3}\left(2j-\frac{2j^2}{j-1}\right)\sqrt{\frac{2}{\pi}}\left[\frac{1}{(1-2j)^3}+\frac{1}{(2j)^3}\right] \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{j-1}\left[\frac{1}{(1-2j)^2}-\frac{1}{(2j)^2}\right] + \frac{2}{3}\frac{j}{j-1}\sqrt{\frac{2}{\pi}}\left[\frac{1}{(1-2j)^3}+\frac{1}{(2j)^3}\right],
\end{aligned}
\tag{B.36}
$$

and the infinite series

$$
\sum_{j\in\complement_{\mathbb{Z}}\{0,1\}} \int_0^\infty ada \int_0^a xdx \int_{-\infty}^0 \frac{\partial}{\partial a} b \left[\frac{\partial}{\partial b}\phi(x-2j(a-b)) - \frac{\partial}{\partial b}\phi(x-2b-2j(a-b))\right] db = \frac{14\pi^2-138}{72\sqrt{2\pi}}.
\tag{B.37}
$$

For summand with $j=1$, we have

$$
\int_0^\infty ada \int_0^a xdx \int_{-\infty}^0 \frac{\partial}{\partial a} b \frac{\partial}{\partial b}\phi(x-2a+2b)db = \mathcal{G}(-2,2) = \frac{5}{12\sqrt{2\pi}}.
\tag{B.38}
$$

Therefore, the joint moment is calculated by Eq. (B.29):

$$
\mathbb{E}[\tilde{u}\tilde{d}|c|] = -2\left(\frac{5}{12\sqrt{2\pi}} + \frac{14\pi^2-138}{72\sqrt{2\pi}}\right) = \frac{54-7\pi^2}{18\sqrt{2\pi}} \approx -0.3344.
\tag{B.39}
$$

Now we can use the results in Eqs. (B.26), (B.27) and (B.39) to calculate the third moments of $(\omega, |c|)$:

$$
\lambda_{1,2} = \mathbb{E}[\omega|c|^2] = \mathbb{E}[\tilde{u}|c|^2] - \mathbb{E}[\tilde{d}|c|^2] = \frac{8}{3}\sqrt{\frac{2}{\pi}} \approx 2.1277,
\tag{B.40}
$$

$$
\begin{aligned}
\lambda_{2,1} &= \mathbb{E}[\omega^2|c|] = \mathbb{E}[\tilde{u}^2|c|] + \mathbb{E}[\tilde{d}^2|c|] - 2\mathbb{E}[\tilde{u}\tilde{d}|c|] \\
&= \frac{17}{3\sqrt{2\pi}} + \frac{1}{3\sqrt{2\pi}} - 2\times\frac{54-7\pi^2}{18\sqrt{2\pi}} = \frac{7}{9}\sqrt{\frac{\pi^3}{2}} \approx 3.0624.
\end{aligned}
\tag{B.41}
$$

Moreover, direct calculation of

$$
\mathbb{E}[|c|^r] = 2\int_0^\infty \frac{x^r}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx
\tag{B.42}
$$

shows the following moments:

$$
\lambda_{0,1} = \sqrt{\frac{2}{\pi}} \approx 0.7979, \qquad \lambda_{0,2} = 1, \qquad \lambda_{0,3} = 2\sqrt{\frac{2}{\pi}} \approx 1.5958, \qquad \lambda_{0,4} = 3.
\tag{B.43}
$$

Appendix C Supplementary Materials

C.1 Monte Carlo Bias Results of Other Estimators

In addition to the Monte Carlo bias results in Section 4.2, Tables C.1 and C.2 report the relative bias (%) in "continuous time" of the TRV estimator of Mancini (2009) and the DV estimator of Andersen et al. (2023), respectively. The choices of truncation parameters for TRV and DV are in line with Andersen et al. (2023).

C.2 Monte Carlo RMSE Results of Other Estimators

In addition to the comparison of finite-sample performances among WV and the main competitors RRV, OWV, RV, TRV and DV, we also consider two return-based IV estimators, i.e., BV of Barndorff-Nielsen and Shephard (2004) and MedRV of Andersen et al. (2012), and also TRV and DV with less aggressive choices of truncation threshold. The RMSE results are shown in Table C.3.

Table C.1: Monte Carlo bias results (%): Truncated realized volatility (TRV)

Panel A: $C_\zeta^{\mathrm{TRV}} = 4$											
	$H = 0$		Gradual Jump			Flash Crash			Gradual Jump with an Intermittent Flash Crash		
Interval	No Jump	Jumps	$\beta = 0.45$	0.35	0.25	$\beta = 0.45$	0.35	0.25	$\beta = 0.45$	0.35	0.25
5 sec	-0.14	0.01	2.29	2.31	1.93	3.63	3.78	3.23	2.85	3.04	2.82
10 sec	-0.13	-0.01	4.03	3.72	3.05	6.31	6.04	5.19	4.74	4.96	4.44
30 sec	-0.09	0.00	9.48	8.16	6.39	16.00	13.94	10.74	11.56	11.53	9.94
1 min	-0.13	0.15	15.81	13.29	10.08	26.64	21.34	15.67	20.38	19.62	16.48
2 min	-0.12	0.19	30.28	24.38	17.82	44.64	37.52	27.78	43.15	37.54	31.50
3 min	-0.18	0.32	33.39	25.94	17.55	63.38	52.32	38.47	37.08	30.29	21.20
5 min	-0.07	0.32	44.10	32.13	20.51	99.42	81.20	58.98	69.29	57.40	43.49
Panel B: $C_\zeta^{\mathrm{TRV}} = 3$											
	$H = 0$		Gradual Jump			Flash Crash			Gradual Jump with an Intermittent Flash Crash		
Interval	No Jump	Jumps	$\beta = 0.45$	0.35	0.25	$\beta = 0.45$	0.35	0.25	$\beta = 0.45$	0.35	0.25
5 sec	-1.57	-1.41	0.53	0.56	0.25	1.66	1.66	1.22	1.07	1.10	0.92
10 sec	-1.67	-1.51	1.86	1.61	1.07	3.75	3.47	2.69	2.61	2.54	2.03
30 sec	-1.76	-1.70	6.09	4.79	3.41	11.02	9.03	6.79	8.33	7.62	5.83
1 min	-1.94	-1.65	10.91	8.46	5.77	19.43	16.63	12.47	15.10	13.31	9.96
2 min	-2.07	-1.93	19.92	15.03	10.26	32.43	25.11	17.07	28.93	24.65	18.77
3 min	-2.30	-1.88	24.93	18.57	12.37	47.77	37.23	25.88	29.14	23.99	16.55
5 min	-2.45	-2.14	36.08	25.92	16.65	77.40	59.28	41.01	48.30	41.49	31.08

Relative biases (%) of the truncated realized volatility (TRV) estimator of Mancini (2009) constructed from 5, 10, 30, 60, 120, 180, and 300-second intervals for 3000 days. The truncation threshold for all returns is with $C_\zeta^{\mathrm{TRV}} = (4, 3)$. The DGP follows the Heston model in Eq. (38), and we follow the persistent noise model of Andersen et al. (2023) to simulate the three different patterns of short-lived explosive trends.

Table C.2: Monte Carlo bias results (%): Differenced-return volatility (DV)

Panel A: $C_\zeta^{\mathrm{DV}} = 4\sqrt{2}$

	$H=0$		Gradual Jump			Flash Crash			Gradual Jump with an Intermittent Flash Crash		
Interval	No Jump	Jumps	$\beta = 0.45$	0.35	0.25	$\beta = 0.45$	0.35	0.25	$\beta = 0.45$	0.35	0.25
5 sec	-0.19	-0.03	-0.13	-0.01	-0.12	0.00	0.23	0.14	0.33	0.13	0.08
10 sec	-0.17	0.03	-0.05	-0.08	-0.12	0.45	0.65	0.78	0.51	0.36	0.36
30 sec	-0.17	-0.15	0.54	-0.34	-0.16	0.44	0.88	1.39	2.40	1.29	1.11
1 min	-0.34	-0.04	1.84	-0.29	-0.39	1.64	3.58	4.86	6.30	3.21	2.45
2 min	-0.64	-0.06	8.66	7.53	1.05	9.66	10.88	9.43	17.95	17.75	11.55
3 min	-0.88	0.01	11.35	2.58	-0.66	16.06	14.42	11.91	21.11	11.37	3.45
5 min	-1.22	-0.24	27.74	11.22	0.60	35.76	28.49	24.22	69.33	55.23	29.91

Panel B: $C_\zeta^{\mathrm{DV}} = 3\sqrt{2}$

	$H=0$		Gradual Jump			Flash Crash			Gradual Jump with an Intermittent Flash Crash		
Interval	No Jump	Jumps	$\beta = 0.45$	0.35	0.25	$\beta = 0.45$	0.35	0.25	$\beta = 0.45$	0.35	0.25
5 sec	-1.61	-1.45	-1.35	-1.08	-1.15	-1.13	-0.87	-0.90	-1.08	-1.11	-1.01
10 sec	-1.67	-1.47	-1.26	-1.08	-1.09	-0.84	-0.50	-0.44	-0.97	-0.82	-0.84
30 sec	-1.80	-1.85	-0.89	-1.13	-0.95	-0.24	0.16	0.35	-0.17	-0.42	-0.60
1 min	-2.04	-1.92	-0.61	-0.96	-1.13	1.11	2.52	2.68	1.00	0.15	-0.09
2 min	-2.47	-2.08	2.49	2.75	-0.24	6.19	6.62	5.26	6.32	6.19	2.83
3 min	-2.75	-2.11	2.07	-1.03	-1.71	5.35	4.38	3.17	8.56	3.75	1.45
5 min	-3.06	-2.91	8.20	0.09	-1.79	9.26	6.29	5.66	25.92	15.67	8.90

Relative biases (%) of the differenced-return volatility (DV) estimator of Andersen et al. (2023) constructed from 5, 10, 30, 60, 120, 180, and 300-second intervals for 3000 days. The truncation threshold for all first-order differenced returns is with $C_\zeta^{\mathrm{DV}} = (4\sqrt{2}, 3\sqrt{2})$. The DGP follows the Heston model in Eq. (38), and we follow the persistent noise model of Andersen et al. (2023) to simulate the three different patterns of short-lived explosive trends.

Table C.3: Monte Carlo RMSE results of other estimators

Panel A: $H = 0$												
	No Jump				With Jumps							
Interval	BV	MedRV	TRV	DV	BV	MedRV	TRV	DV				
1 min	0.74	0.78	0.65	0.79	0.93	0.79	0.65	0.80				
2 min	1.04	1.12	0.91	1.11	1.36	1.20	0.92	1.13				
3 min	1.27	1.36	1.12	1.37	1.65	1.44	1.14	1.40				
5 min	1.66	1.78	1.45	1.76	2.10	1.86	1.49	1.85				
10 min	2.34	2.52	2.04	2.47	3.00	2.75	2.23	2.85				
Panel B: Gradual Jump												
	$\beta = 0.45$				$\beta = 0.35$				$\beta = 0.25$			
Interval	BV	MedRV	TRV	DV	BV	MedRV	TRV	DV	BV	MedRV	TRV	DV
1 min	3.85	2.87	1.59	0.86	4.50	2.77	1.39	0.79	5.24	2.46	1.15	0.79
2 min	6.79	6.25	2.97	1.53	7.65	5.99	2.46	1.39	8.63	5.18	2.00	1.21
3 min	8.79	6.39	3.25	1.99	9.06	5.43	2.66	1.66	9.22	4.40	2.03	1.46
5 min	12.80	9.16	4.36	3.75	12.49	7.47	3.41	2.96	12.00	5.74	2.54	2.26
10 min	21.46	20.32	15.66	8.79	20.60	16.42	9.55	10.12	19.21	11.99	5.51	8.48
Panel C: Flash Crash												
	$\beta = 0.45$				$\beta = 0.35$				$\beta = 0.25$			
Interval	BV	MedRV	TRV	DV	BV	MedRV	TRV	DV	BV	MedRV	TRV	DV
1 min	5.13	4.78	2.52	0.82	5.72	5.33	2.04	0.86	5.75	5.48	1.59	0.92
2 min	5.50	5.74	4.16	1.48	4.77	4.99	3.56	1.52	3.72	3.81	2.76	1.47
3 min	8.41	7.92	5.89	2.23	7.32	6.75	4.96	2.14	5.64	5.09	3.80	2.00
5 min	13.28	12.15	9.18	4.22	11.18	10.19	7.68	3.84	8.29	7.52	5.73	3.44
10 min	25.65	24.33	18.56	13.72	21.13	20.20	15.52	11.98	15.11	14.68	11.39	9.78
Panel D: Gradual Jump with an Intermittent Flash Crash												
	$\beta = 0.45$				$\beta = 0.35$				$\beta = 0.25$			
Interval	BV	MedRV	TRV	DV	BV	MedRV	TRV	DV	BV	MedRV	TRV	DV
1 min	3.69	3.00	1.94	1.04	4.44	3.13	1.89	0.90	5.19	2.91	1.66	0.87
2 min	6.62	6.11	4.09	2.09	7.62	6.30	3.54	2.11	8.56	5.75	3.06	1.77
3 min	7.43	5.86	3.56	2.56	7.90	5.25	3.02	2.05	8.14	4.26	2.31	1.50
5 min	13.04	10.05	6.75	6.82	13.31	9.09	5.59	6.23	12.84	7.57	4.45	4.07
10 min	19.92	17.86	17.41	13.37	20.31	16.50	15.10	15.37	19.52	13.62	8.94	15.01

RMSEs (multiplied by 10^5) of different IV estimators for 10000 days. We follow the persistent noise model of Andersen et al. (2023) to simulate the three different patterns of episodic extreme return persistence. All range- and return-based estimators are constructed from the simulated 1, 2, 3, 5, and 10-minute candlesticks. The choice of truncation parameters for TRV and DV follows the instructions in Section 4.2, with $(C_\zeta^{\mathrm{TRV}}, C_\zeta^{\mathrm{DV}}, \varpi) = (4, 4\sqrt{2}, 0.49)$.

C.3 Monte Carlo RMSE Results with Market Microstructure Noise

To evaluate the impact of market microstructure noise on the finite-sample performance of WV, we augment the Heston model in Eq. (38) with an additive heterogeneous Gaussian noise term for the simulation of all "transactions" (Euler steps), which follows the Monte Carlo simulations in Aït-Sahalia et al. (2012) and Christensen et al. (2022):

$$Y_i = X_i + \epsilon_i, \quad \epsilon_i \sim \mathcal{N}(0, \omega_i^2), \quad \text{where } \omega_i = 2\frac{\sigma_i}{\sqrt{N}}, \tag{C.1}$$

for $i = 0, 1, \ldots, N$. Besides the additive noise, we consider the rounding errors on the price level, i.e., let the observed prices $e^{Y_i} = e^{X_i+\epsilon_i}$ be further rounded to cents. The observed logarithmic prices are given as

$$Y_i = \ln\left(\left[\frac{e^{X_i+\epsilon_i}}{0.01}\right] \times 0.01\right), \tag{C.2}$$

where the function $[x]$ rounds a number x to the nearest integer. We maintain the same parameter choices as specified in Section 4.1.

We construct the WV estimator from both 1, 5, and 10-minute candlesticks. Apart from the WV that requires only OHLCs for sparse intervals, we also consider established noise-corrected estimators that utilize all noise-contaminated transactions, such as the pre-averaged realized volatility (PRV) and bipower variation (PBV), as detailed in Jacod et al. (2009), Podolskij and Vetter (2009), and Christensen et al. (2025). With all Euler discretization steps available, we replicate the ideal scenario for a limited number of practitioners with access to ultra-high-frequency data.

Table C.4 reports the RMSEs of all selected IV estimators when there exists the heterogeneous Gaussian noise in Eq. (C.1) at the tick level. The exaggerated RMSEs of the tick-level RV show the impact of the simulated noise. For the WV based on only intraday candlesticks over sparsely sampled intervals, it exhibits elevated RMSEs compared to the noise-free case (Table 2) when the interval length is relatively small (1 minute). When the WV is constructed from 5-minute and 10-minute candlesticks, the RMSEs results are close or nearly identical to the noise-free case. When there exists no extreme price movement, the tick-level PRV achieves the smallest RMSE, which is consistent with the theoretical results under infill asymptotics, see, e.g., Theorem 1 in Christensen et al. (2025). When jumps occur approximately once per week ($\lambda = 1/5$), the tick-level PBV shows the smallest RMSE, which verifies its robustness to discontinuities with noise-contaminated observations. In both cases with $H = 0$, the WV shows nearly identical, small RMSEs, which highlights the advantage of utilizing readily available intraday candlesticks. In the presence of short-lived explosive trends, the WV constructed from one-minute candlesticks consistently achieves the smallest RMSEs across all three scenarios.

Table C.4: Monte Carlo RMSE results with market microstructure noise

			$H=0$		Gradual Jump	Flash Crash	Gradual Jump with an Intermittent Flash Crash
$\widehat{V}$	Data Type	Granularity	No Jumps	With Jumps	$\beta=0.35$	$\beta=0.35$	$\beta=0.35$
RV	Transactions	Tick	74.41	76.43	77.29	78.63	77.41
WV(3)	Candlesticks	1 min	1.00	1.01	0.81	0.70	0.75
		5 min	0.98	0.98	0.87	1.04	0.89
		10 min	1.30	1.31	1.27	4.72	1.29
WV(2)	Candlesticks	1 min	0.93	0.94	0.80	0.70	0.75
		5 min	0.86	0.86	0.87	0.92	0.88
		10 min	1.14	1.15	1.25	1.96	1.28
PRV	Transactions	Tick	0.17	6.80	6.83	7.12	6.58
		1 sec	0.88	6.98	9.60	8.83	8.87
		5 sec	0.81	6.91	12.19	10.99	10.86
PBV	Transactions	Tick	0.18	0.25	1.03	1.47	1.12
		1 sec	0.88	1.13	4.73	5.12	4.46
		5 sec	0.87	1.20	7.02	7.45	6.48

RMSEs (multiplied by 10^5) of different IV estimators for 10000 days. We follow the persistent noise model of Andersen et al. (2023) to simulate the three different patterns of short-lived explosive trends. The WV estimators are constructed from the simulated 1, 5, and 10-minute candlesticks. The choice of truncation parameters for WV follows the instructions in Section 4.2, with $C_\zeta = 3$ and 2 (represented by WV(3) and WV(2), respectively). The return-based estimators RV, PRV, and PBV are constructed from the simulated ultra-high-frequency data at the tick level or under equidistant sampling at high granularity levels. The pre-averaging window is $k_n = \lceil \theta\sqrt{N} \rceil$ with $\theta = 0.5$.

C.4 Empirical Results: Additional Comparisons

Practitioners with access to the complete TAQ dataset can construct noise-robust IV estimators with all tick-level transaction data. Table C.5 presents additional empirical results on volatility forecasting to Table 4. With the same one-day-ahead forecasting targets, we compare the empirical performance of HAR models with two sets of RHS predictors: (i) range-based estimators based on summary information provided by 5-minute candlesticks, and (ii) the PRV and PBV estimators constructed from full tick-level data. The out-of-sample performance across two subsets of days is reported in Table C.6. Our findings reveal that the simple HAR-WV model achieves nearly the same level of predictability as HAR models augmented with pre-averaged estimators based on tick-level transaction data. Furthermore, the smaller MSE results of HAR-WV on days following extreme price movements highlight the advantages of robustness in improving volatility forecasting accuracy. Most importantly, the candlestick-based estimators provide significant practical benefits, including ease of computation and dramatically lower requirement of data availability, which makes it a more accessible and efficient option for general investors.

Table C.5: Daily out-of-sample 5-minute HAR volatility forecasts

	RV		TRV		WV	
	MSE	QLIKE	MSE	QLIKE	MSE	QLIKE
Panel A: RW Forecasts						
HAR-RRV	2.38	0.35	1.47	0.36	0.90	0.25
HAR-OKV	2.19	0.35	1.35	0.36	0.83	0.25
HAR-WV	2.14	0.34	1.32	0.36	0.83	0.24
HAR-HWV	2.12	0.34	1.30	0.36	0.82	0.24
SHAR-WV	2.50	0.36	1.61	0.37	0.94	0.25
HARQ-WV	1.94	0.33	1.35	0.35	0.96	0.25
HAR-PRV	2.08	0.34	1.30	0.35	0.79	0.24
HAR-PBV	2.10	0.34	1.31	0.36	0.80	0.24
Panel B: EW Forecasts						
HAR-RRV	1.76	0.33	1.08	0.35	0.66	0.24
HAR-OKV	1.64	0.32	1.00	0.33	0.61	0.23
HAR-WV	1.56	0.30	0.95	0.31	0.58	0.21
HAR-HWV	1.60	0.30	0.97	0.31	0.60	0.21
SHAR-WV	1.59	0.32	0.99	0.32	0.59	0.21
HARQ-WV	1.53	0.28	0.98	0.29	0.59	0.19
HAR-PRV	1.58	0.30	0.97	0.31	0.59	0.21
HAR-PBV	1.59	0.30	0.99	0.31	0.60	0.21

MSE ($\times 10^8$) and QLIKE of daily out-of-sample volatility forecasts for the SPDR S&P 500 ETF Trust (SPY). The HAR model is re-estimated via OLS with both rolling windows and expanding windows, respectively. All range-based estimators are constructed from 5-minute candlesticks. The PRV and PBV estimators are constructed from all transaction prices at tick level. The pre-averaging window is $k_n = \lceil \theta\sqrt{N} \rceil$ with $\theta = 0.5$.

Table C.6: Daily out-of-sample 5-minute HAR volatility forecasts on different subsets of days

	RV				TRV				WV			
Trading Days	EPM + 1		Other		EPM + 1		Other		EPM + 1		Other	
	MSE	QLIKE	MSE	QLIKE	MSE	QLIKE	MSE	QLIKE	MSE	QLIKE	MSE	QLIKE
Panel A: RW Forecasts												
HAR-RRV	4.59	0.39	2.01	0.35	2.77	0.43	1.25	0.35	1.29	0.29	0.84	0.24
HAR-OKV	3.48	0.38	1.97	0.34	2.17	0.42	1.22	0.35	0.98	0.28	0.81	0.24
HAR-WV	2.53	0.33	2.08	0.35	1.63	0.35	1.31	0.36	0.71	0.23	0.85	0.25
HAR-HWV	2.46	0.30	2.06	0.36	1.58	0.32	1.27	0.37	0.69	0.21	0.84	0.25
SHAR-WV	3.35	0.37	2.36	0.37	2.23	0.40	1.51	0.37	0.90	0.25	0.94	0.25
HARQ-WV	1.28	0.31	2.05	0.33	0.76	0.37	1.45	0.34	0.48	0.25	1.04	0.25
HAR-PRV	3.38	0.36	1.87	0.33	2.10	0.39	1.17	0.34	0.93	0.26	0.76	0.23
HAR-PBV	3.41	0.36	1.87	0.33	2.11	0.39	1.18	0.34	0.94	0.26	0.77	0.23
Panel B: EW Forecasts												
HAR-RRV	3.04	0.39	1.55	0.32	1.74	0.42	0.97	0.34	0.81	0.28	0.64	0.23
HAR-OKV	2.36	0.38	1.52	0.31	1.39	0.41	0.94	0.32	0.62	0.27	0.61	0.22
HAR-WV	1.89	0.31	1.50	0.30	1.11	0.29	0.92	0.31	0.48	0.21	0.60	0.22
HAR-HWV	1.84	0.27	1.56	0.30	1.11	0.29	0.95	0.32	0.46	0.19	0.62	0.21
SHAR-WV	1.73	0.34	1.57	0.31	1.07	0.36	0.98	0.32	0.46	0.23	0.61	0.21
HARQ-WV	1.55	0.30	1.53	0.26	0.92	0.31	0.99	0.27	0.37	0.21	0.63	0.19
HAR-PRV	2.23	0.36	1.48	0.29	1.31	0.38	0.92	0.30	0.58	0.25	0.59	0.20
HAR-PBV	2.23	0.36	1.49	0.29	1.31	0.38	0.93	0.30	0.58	0.25	0.60	0.20

MSE ($\times 10^8$) and QLIKE of daily out-of-sample volatility forecasts for the SPDR S&P 500 ETF Trust (SPY) on (i) days that follow extreme price movements identified on preceding days ("After EPM"), and (ii) all remaining days. The existence of jumps or excessive return drift is identified with the Hausman test in Eq. (35). The HAR model is re-estimated via OLS in rolling windows and expanding windows, respectively. All range-based estimators are constructed from 5-minute candlesticks. The PRV and PBV estimators are constructed from the all transaction prices at tick level. The pre-averaging window is $k_n = \lceil \theta\sqrt{N} \rceil$ with $\theta = 0.5$.

C.5 Empirical Results: Model Confidence Sets

Table C.7 reports the MCS p-values for our selection of volatility models at two confidence levels, 75% and 90%, based on both loss functions MSE and QLIKE.

Table C.7: MCS p-values for daily out-of-sample 5-minute HAR volatility forecasts

	RV		TRV		WV	
	MSE	QLIKE	MSE	QLIKE	MSE	QLIKE
Panel A: RW Forecasts						
HAR-RV	0.639**	0.155*	0.562**	0.185*	0.550**	0.078
HAR-BV	0.671**	0.124*	0.635**	0.185*	0.573**	0.079
HAR-MedRV	0.671**	0.182*	0.711**	0.268**	0.712**	0.113*
HAR-TRV	0.569**	0.437**	0.557**	0.822**	0.526**	0.988**
HAR-DV	0.561**	0.386**	0.562**	0.576**	0.672**	0.924**
HAR-RRV	0.623**	0.410**	0.557**	0.739**	0.526**	0.809**
HAR-OKV	0.760**	0.437**	0.964**	0.822**	0.946**	0.782**
HAR-WV	0.761**	0.437**	0.964**	0.822**	0.946**	1.000**
HAR-HBV	0.639**	0.297**	0.581**	0.471**	0.578**	0.224*
HAR-HDV	0.569**	0.437**	0.557**	0.822**	0.526**	0.988**
HAR-HWV	0.761**	0.437**	1.000**	0.822**	1.000**	0.988**
SHAR-RV	0.561**	0.124*	0.557**	0.283**	0.526**	0.412**
SHAR-WV	0.623**	0.117*	0.964**	0.283**	0.946**	0.876**
HARQ-RV	0.761**	0.647**	0.711**	0.957**	0.943**	0.877**
HARQ-WV	1.000**	1.000**	0.964**	1.000**	0.943**	0.972**
Panel B: EW Forecasts						
HAR-RV	0.707**	0.000	0.553**	0.000	0.490**	0.000
HAR-BV	0.698**	0.000	0.592**	0.000	0.536**	0.000
HAR-MedRV	0.782**	0.000	0.717**	0.000	0.634**	0.000
HAR-TRV	0.831**	0.009	0.853**	0.003	0.647**	0.000
HAR-DV	0.583**	0.004	0.553**	0.002	0.561**	0.000
HAR-RRV	0.716**	0.000	0.557**	0.000	0.487**	0.000
HAR-OKV	0.739**	0.000	0.668**	0.000	0.535**	0.000
HAR-WV	0.868**	0.014	1.000**	0.004	0.647**	0.005
HAR-HBV	0.729**	0.000	0.612**	0.000	0.555**	0.000
HAR-HDV	0.735**	0.009	0.717**	0.003	0.647**	0.003
HAR-HWV	0.768**	0.010	0.853**	0.012	0.634**	0.013
SHAR-RV	0.376**	0.000	0.477**	0.000	0.485**	0.000
SHAR-WV	0.868**	0.000	0.929**	0.000	1.000**	0.000
HARQ-RV	0.831**	0.956**	0.853**	0.691**	0.647**	0.012
HARQ-WV	1.000**	1.000**	0.887**	1.000**	0.647**	1.000**

MCS p-values for our selection of volatility models. p-values marked with * and ** are included in MCS at confidence levels 75% and 90%, respectively. We consider both MSE and QLIKE as loss functions, and estimate the bootstrap distributions of the range test statistics, each with 5,000 replications and a block size of $\lfloor \sqrt[3]{M} \rfloor = 13$ days.

In addition to Table C.7, we conduct two MCS exercises on selected subsets of models. Table C.8 reports the MCS based on both MSE and QLIKE for all simple HAR models estimated with

(i) Return-based estimators: RV, BV, MedRV, TRV, and DV;
(ii) Range-based estimators: RRV, OKV, and WV.

Table C.9 extends the subset by adding

(iii) Hybrid estimators in the form of Eq. (49): HBV, HDV, and HWV.

These supplementary MCS results are restricted to simple HAR models. All HAR extensions such as HARQ and SHAR are excluded.

Table C.8: MCS p-values for daily out-of-sample 5-minute simple HAR volatility forecasts

	RV		TRV		WV	
	MSE	QLIKE	MSE	QLIKE	MSE	QLIKE
Panel A: RW Forecasts						
HAR-RV	0.376**	0.086	0.387**	0.116*	0.346**	0.037
HAR-BV	0.378**	0.084	0.445**	0.116*	0.346**	0.042*
HAR-MedRV	0.378**	0.106*	0.445**	0.163*	0.508**	0.055*
HAR-TRV	0.348**	0.751**	0.372**	0.961**	0.327**	0.933**
HAR-DV	0.348**	0.729**	0.445**	0.617**	0.508**	0.649**
HAR-RRV	0.348**	0.693**	0.372**	0.617**	0.327**	0.500**
HAR-OKV	0.702**	0.751**	0.998**	0.961**	0.977**	0.500**
HAR-WV	1.000**	1.000**	1.000**	1.000**	1.000**	1.000**
Panel B: EW Forecasts						
HAR-RV	0.439**	0.000	0.373**	0.000	0.490**	0.000
HAR-BV	0.439**	0.000	0.434**	0.000	0.536**	0.000
HAR-MedRV	0.439**	0.000	0.434**	0.000	0.634**	0.000
HAR-TRV	0.439**	0.033	0.434**	0.040	0.647**	0.001
HAR-DV	0.357**	0.015	0.434**	0.017	0.561**	0.001
HAR-RRV	0.439**	0.000	0.434**	0.000	0.487**	0.000
HAR-OKV	0.439**	0.000	0.434**	0.000	0.535**	0.000
HAR-WV	1.000**	1.000**	1.000**	1.000**	0.647**	1.000**

MCS p-values for our selection of simple HAR models. p-values marked with * and ** are included in MCS at confidence levels 75% and 90%, respectively. We consider both MSE and QLIKE as loss functions, and implement the bootstrap with 5,000 replications for each pairwise test and a block size of $\lfloor \sqrt[3]{M} \rfloor = 13$ days.

Table C.9: MCS p-values for daily out-of-sample 5-minute simple HAR volatility forecasts

	RV		TRV		WV	
	MSE	QLIKE	MSE	QLIKE	MSE	QLIKE
Panel A: RW Forecasts						
HAR-RV	0.550**	0.113*	0.465**	0.158*	0.441**	0.058
HAR-BV	0.550**	0.093	0.465**	0.158*	0.449**	0.072
HAR-MedRV	0.550**	0.138*	0.502**	0.235*	0.499**	0.087
HAR-TRV	0.433**	0.880**	0.452**	0.977**	0.407**	0.990**
HAR-DV	0.428**	0.835**	0.465**	0.740**	0.499**	0.893**
HAR-RRV	0.452**	0.747**	0.447**	0.746**	0.407**	0.674**
HAR-OKV	0.815**	0.880**	0.831**	0.977**	0.865**	0.667**
HAR-WV	1.000**	1.000**	1.000**	1.000**	1.000**	1.000**
HAR-HBV	0.550**	0.361**	0.465**	0.459**	0.449**	0.171*
HAR-HDV	0.433**	0.562**	0.452**	0.847**	0.407**	0.990**
HAR-HWV	1.000**	0.880**	1.000**	1.000**	1.000**	0.990**
Panel B: EW Forecasts						
HAR-RV	0.569**	0.000	0.456**	0.000	0.359**	0.000
HAR-BV	0.562**	0.000	0.484**	0.000	0.380**	0.000
HAR-MedRV	0.570**	0.000	0.521**	0.000	0.486**	0.000
HAR-TRV	0.570**	0.083	0.567**	0.108*	0.465**	0.001
HAR-DV	0.442**	0.030	0.521**	0.004	0.412**	0.000
HAR-RRV	0.570**	0.000	0.484**	0.000	0.372**	0.000
HAR-OKV	0.570**	0.003	0.521**	0.000	0.420**	0.000
HAR-WV	1.000**	1.000**	1.000**	0.945**	1.000**	0.836**
HAR-HBV	0.570**	0.000	0.491**	0.000	0.420**	0.000
HAR-HDV	0.570**	0.083	0.521**	0.108*	0.581**	0.001
HAR-HWV	0.570**	0.953**	0.567**	1.000**	0.581**	1.000**

MCS p-values for our selection of simple HAR models. p-values marked with * and ** are included in MCS at confidence levels 75% and 90%, respectively. We consider both MSE and QLIKE as loss functions, and implement the bootstrap with 5,000 replications for each pairwise test and a block size of $\lfloor \sqrt[3]{M} \rfloor = 13$ days.

References

Aït-Sahalia, Y. and Jacod, J. (2009). Testing for jumps in a discretely observed process. *Annals of Statistics*, 37(1):184–222.

Aït-Sahalia, Y., Jacod, J., and Li, J. (2012). Testing for jumps in noisy high frequency data. *Journal of Econometrics*, 168(2):207–222.

Andersen, T. G., Dobrev, D., and Schaumburg, E. (2012). Jump-robust volatility estimation using nearest neighbor truncation. *Journal of Econometrics*, 169(1):75–93.

Andersen, T. G., Li, Y., Todorov, V., and Zhou, B. (2023). Volatility measurement with pockets of extreme return persistence. *Journal of Econometrics*, 237(2):105439.

Barndorff-Nielsen, O. E. and Shephard, N. (2004). Power and bipower variation with stochastic volatility and jumps. *Journal of Financial Econometrics*, 2(1):1–37.

Christensen, K., Oomen, R., and Renò, R. (2022). The drift burst hypothesis. *Journal of Econometrics*, 227(2):461–497.

Christensen, K. and Podolskij, M. (2007). Realized range-based estimation of integrated variance. *Journal of Econometrics*, 141(2):323–349.

Christensen, K. and Podolskij, M. (2012). Asymptotic theory of range-based multipower variation. *Journal of Financial Econometrics*, 10(3):417–456.

Christensen, K., Timmermann, A., and Veliyev, B. (2025). Warp speed price moves: Jumps after earnings announcements. *Journal of Financial Economics*, 167:104010.

Duembgen, M. and Podolskij, M. (2015). High-frequency asymptotics for path-dependent functionals of Itô semimartingales. *Stochastic Processes and their Applications*, 125(4):1195–1217.

Feller, W. (1951). The asymptotic distribution of the range of sums of independent random variables. *Annals of Mathematical Statistics*, 22(3):427–432.

Garman, M. B. and Klass, M. J. (1980). On the estimation of security price volatilities from historical data. *Journal of Business*, 53(1):67–78.

Jacod, J., Li, Y., Mykland, P. A., Podolskij, M., and Vetter, M. (2009). Microstructure noise in the continuous case: The pre-averaging approach. *Stochastic Processes and their Applications*, 119(7):2249–2276.

Jacod, J., Li, Y., and Zheng, X. (2019). Estimating the integrated volatility with tick observations. *Journal of Econometrics*, 208(1):80–100.

Jacod, J. and Protter, P. (2012). *Discretization of Processes*. Springer.

Laurent, S., Renò, R., and Shi, S. (2024). Realized drift. *Journal of Econometrics*, forthcoming.

Liao, Y. and Todorov, V. (2024). Changes in the span of systematic risk exposures. *Quantitative Economics*, 15(3):817–847.

Mancini, C. (2009). Non-parametric threshold estimation for models with stochastic diffusion coefficient and jumps. *Scandinavian Journal of Statistics*, 36(2):270–296.

Meilijson, I. (2011). The Garman-Klass volatility estimator revisited. *Revstat*, 9(3):199–212.

Parkinson, M. (1980). The extreme value method for estimating the variance of the rate of return. *Journal of Business*, 53(1):61–65.

Podolskij, M. and Vetter, M. (2009). Bipower-type estimation in a noisy diffusion setting. *Stochastic Processes and their Applications*, 119(9):2803–2831.